

Extending a system in the calculus of structures with a self-dual quantifier

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Abstract

We recall that SBV, a proof system developed under the methodology of deep inference, extends multiplicative linear logic with the self-dual non-commutative logical operator *Seq*. We introduce SBVQ that extends SBV by adding the self-dual quantifier *Sdq*. The system SBVQ is consistent because we prove that (the analogous of) cut elimination holds for it. Its new logical operator *Sdq* operationally behaves as a binder, in a way that the interplay between *Seq*, and *Sdq* can model β -reduction of linear λ -calculus inside the cut-free subsystem BVQ of SBVQ. The long term aim is to keep developing a programme whose goal is to give pure logical accounts of computational primitives under the proof-search-as-computation analogy, by means of minimal, and incremental extensions of SBV.

1 Introduction.

This is a work in structural proof-theory. We extend SBV [5], the paradigmatic system of the deep inference methodology to design proof systems.

Deep inference (DI). One of the main aspects of DI is that logical systems can be designed as they were rewriting systems, namely, systems with rules that apply *deeply* inside terms, or, equivalently, in any suitable context. We must read “deep” as opposed to “shallow”. Rules of sequent and natural deduction systems are shallow because they build proofs whose form mimics the one of formulas. Thanks to the deep application of its rules, BV substantially extends multiplicative linear logic (MLL) [3] with the non commutative binary operator *Seq*, whose logical properties are strictly connected to the expressiveness of BV itself. Any limits we might put on the application depth of BV rules would yield a strictly less expressive system [16] indeed. An extension of BV, by means of linear logic exponentials [6, 7, 8, 15] is NEL, whose provability is undecidable [13].

Contributions, and motivations. We introduce SBVQ. It is SBV plus a quantifier that we identify as *Sdq*, which abbreviates “Self-dual quantifier”. The relevant feature of *Sdq* is to bind variable names of SBVQ only. The consequence is twofold. First, we do not need to classify *Sdq* as either an existential, or a universal quantifier. Indeed, binding variable names

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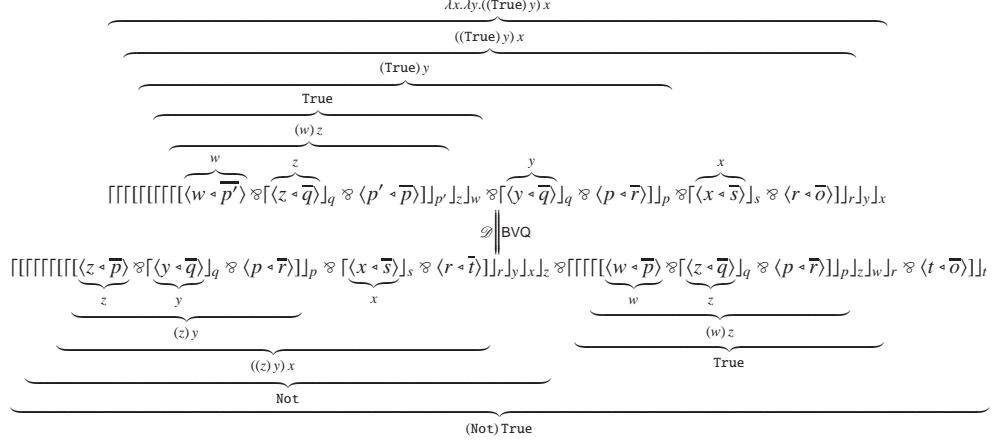


Figure 1: Computing λ -term (Not) True in BVQ.

only, it never requires to distinguish if the quantification is over a variable which we can think of as an assumption or as a conclusion. Hence, a second consequence is that Sdq naturally becomes self-dual. So, SBVQ can be viewed as a minimal extension of SBVQ by means of a logical operator whose instances identify regions of formulas where specific variable names can essentially change freely.

The work may be viewed as divided in two parts. The first is about proving that SBVQ is consistent. Namely, SBVQ enjoys Splitting (Section 3) which identifies the subset BVQ of SBVQ which plays the role of cut-free fragment.

The second part of the work gives to Sdq an operational semantics. Exploiting that Sdq is a binder, we show that its interplay with Seq makes proof-search inside BVQ complete w.r.t. the basic functional computation expressed by linear λ -calculus. We recall that functions linear λ -calculus represents use their arguments exactly once in the course of the evaluation. So, the set of functions it can express is quite limited, but large enough to let the decision about which is the normal form of two linear λ -terms a *polynomial time complete* problem [10]. Completeness amounts to first defining an embedding $\llbracket \cdot \rrbracket$ from linear λ -terms to formulas of BVQ (Section 5.) Then, completeness states that, for every linear λ -term M , and every atom o , which plays the role of an output-channel, if M reduces to N , then there is a derivation \mathcal{D} of BVQ , that derives the conclusion $\llbracket M \rrbracket_o$ from the assumption $\llbracket N \rrbracket_o$. (Theorem 5.1.) For example, let us recall a possible encoding of boolean values, and of boolean negation:

$\text{Not} \equiv \lambda z.\lambda x.\lambda y.((z)y)x$	$\text{True} \equiv \lambda w.\lambda z.(w)z$	$\text{False} \equiv \lambda w.\lambda z.(z)w$
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Figure 1 shows (part of) a non trivial example of completeness. We have a derivation of BVQ whose conclusion encodes (Not) True, while the premise encodes its β -reduct $\lambda x.\lambda y.((\text{True})y)x$.

Finally, showing completeness means we keep developing a programme whose goal is to give pure logical accounts of computational primitives under the proof-search-as-computation analogy, by means of minimal extensions of SBV . This programme begins in [2]. It shows that Seq soundly, and completely models CC_{sp} , the restriction of Milner CCS [?] to a fragment that contains sequential, and parallel composition only.

Related works. This work directly relates to [11], and [12] as follows. First here we choose a better terminology. Current “self-dual quantifier” Sdq were dubbed as “renaming” in both [11], and [12], putting too much emphasis about its operational meaning. Moreover, this work (i) cleans up the definition, and the properties of Sdq , (ii) generalizes the statements of some principal property, correcting non-crucial flows in their proofs, (iii) states and proves deduction and standardization properties, (iv) includes details of many proofs of the given statements, (v) simplifies the map $\langle \cdot \rangle$ from linear λ -terms to formulas of BVQ dropping any reference to explicit substitutions inside linear λ -terms, which was, instead, mandatory in [11, 12], (vi) among the conclusions (Section 6), anticipates that BVQ can be complete, and not only sound, w.r.t. a suitable extension of the above fragment CCS of Milner CCS.

Besides [2], further related works are [1, 4], and [17].

The former restates natural deduction of the negative fragment of intuitionistic logic into a DI system from which extracting an algebra of combinators where interpreting λ -terms. So, the connection is the aim of giving a computational interpretation to a DI system. Further investigation on the computational nature of deep inference system is in [4]. It shows relations among lambda calculi with explicit substitutions, and intuitionistic systems redefined in accordance with the deep inference approach to proof theory. Specifically, [4] shows the impact on the design of λ -calculi with explicit substitutions of intuitionistic logic reworked in terms of nested sequents or of calculus of structures, under both the proofs-as-programs, and formulas-as-programs paradigms.

Finally, [17] inspired the two-arguments map $\langle \cdot \rangle$ from linear λ -terms to formulas of BVQ. Anticipating a bit the content of Section 4, the definition of the basic clause of $\langle \cdot \rangle$ is $\langle x \rangle_o = \langle x \circ \bar{o} \rangle$. Intuitively, the linear λ -calculus variable x in $\langle \cdot \rangle$ becomes the name of an input channel to the left of the occurrence \circ of Seq . The input channel is forwarded to the output channel o in analogy with the *forwarder* $[x]_o = x(\circ) . \bar{o}(\circ)$ which comes from [9], and which is one of the defining clauses of the *output-based embedding* of *standard* λ -calculus with *explicit substitutions* into π -calculus [17]. So, SBV can model a forwarder, the basic input/output communication flow that λ -variables realize. The introduction of Sdq allows to model any *on-the-fly renaming* of channels that serves to model the substitution of a term for a bound variable, namely, the linear β -reduction process of linear λ -calculus.

Road map. Section 2 introduces the extension SBVQ (BVQ) of SBV (BV). Section 3 proves that SBVQ is consistent by extending the proof of (the analogous of) cut-elimination for SBV to SBVQ. Section 4 recalls linear λ -calculus, and defines the embedding of its terms to formulas of BVQ. Section 5 shows the completeness of BVQ w.r.t. linear λ -calculus, namely it shows that every computation in the latter corresponds to a proof-search in the former. Section 6 comments about the lack of a reasonable soundness of BVQ w.r.t. to λ -calculus, and points to future work.

2 Systems SBVQ and BVQ

We recall and clean-up the definitions of [11, 12].

Structures. Let a, b, c, \dots denote the elements of a countable set of *positive propositional variables*. Let $\bar{a}, \bar{b}, \bar{c}, \dots$ denote the elements of a countable set of *negative propositional variables*. The set of *names*, which we range over by l, m , and n , contains both positive, and negative propositional variables, and nothing else. Let \circ be a constant, different from any name, which we call *unit*. The set of *atoms* contains both names and the unit, while the set

of *structures* identifies formulas of SBV. Structures belong to the language of the grammar in (1).

$$R ::= \circ \mid \mathbf{1} \mid \bar{R} \mid (R \otimes R) \mid \langle R \ast R \rangle \mid [R \wp R] \mid \lceil R \rceil_a \quad (1)$$

We use K, P, R, T, U, V to range over structures. As in SBV, \bar{R} is a Not structure, $(R \otimes T)$ is a CoPar structure, $\langle R \ast T \rangle$ is a Seq structure, and $[R \wp T]$ is a Par structure. The Sdq structure $\lceil R \rceil_a$ is new. It comes with the proviso that a must be a positive atom. Namely, $\lceil R \rceil_{\bar{a}}$ is not in the syntax. Sdq induces notions of *free*, and *bound names*, defined in (2).

$$\begin{array}{ll} \{a\} = \text{fn}(a) \cup \text{fn}(\bar{a}) & \emptyset = \text{bn}(a) \cup \text{bn}(\bar{a}) \\ a \in \text{fn}(\bar{R}) \text{ if } a \in \text{fn}(R) & a \in \text{bn}(\bar{R}) \text{ if } a \in \text{bn}(R) \\ a \in \text{fn}((R \otimes T)) \text{ if } a \in \text{fn}(R) \cup \text{fn}(T) & a \in \text{bn}(\langle R \ast T \rangle) \text{ if } a \in \text{bn}(R) \cup \text{bn}(T) \\ a \in \text{fn}(\langle R \ast T \rangle) \text{ if } a \in \text{fn}(R) \cup \text{fn}(T) & a \in \text{bn}((R \otimes T)) \text{ if } a \in \text{bn}(R) \cup \text{bn}(T) \\ a \in \text{fn}([R \wp T]) \text{ if } a \in \text{fn}(R) \cup \text{fn}(T) & a \in \text{bn}([R \wp T]) \text{ if } a \in \text{bn}(R) \cup \text{bn}(T) \\ a \in \text{fn}(\lceil R \rceil_b) \text{ if } a \neq b \text{ and } a \in \text{fn}(R) & a \in \text{bn}(\lceil R \rceil_b) \text{ if } a \equiv b \text{ or } a \in \text{bn}(R) \end{array} \quad (2)$$

Finally, (3) defines the substitution $R\{^a/b\}$ that replaces (i) the atom a for the free occurrences of b , and (ii) the atom \bar{a} for those ones of \bar{b} , in R .

$$\begin{array}{lll} \circ\{^a/b\} \equiv \circ & c\{^a/b\} \equiv c & \langle R \ast T \rangle\{^a/b\} \equiv \langle R\{^a/b\} \ast T\{^a/b\} \rangle \\ b\{^a/b\} \equiv a & \bar{c}\{^a/b\} \equiv \bar{c} & [R \wp T]\{^a/b\} \equiv [R\{^a/b\} \wp T\{^a/b\}] \\ \bar{b}\{^a/b\} \equiv \bar{a} & \bar{R}\{^a/b\} \equiv \overline{R\{^a/b\}} & \lceil R \rceil_b\{^a/b\} \equiv \lceil R \rceil_b \\ (R \otimes T)\{^a/b\} \equiv (R\{^a/b\} \otimes T\{^a/b\}) & & \lceil R \rceil_c\{^a/b\} \equiv \lceil R\{^a/b\} \rceil_c \end{array} \quad (3)$$

Size of the structures. The *size* $|R|$ of R is the number of occurrences of atoms in R plus the number of occurrences of Sdq that effectively bind an atom.

Example 2.1 (Size of the structures) We have $\|[a \wp \bar{a}]\| = \|[a \wp \bar{a}]\|_b = 2$ for we do not count the occurrence of $\lceil \cdot \rceil$. Instead, we count it in $\lceil [a \wp \bar{a}] \rceil_a$, getting $\|[a \wp \bar{a}]\|_a = 3$.

(Structure) Contexts. We denote them by $S\{ \}$. A context is a structure with a single hole $\{ \}$ in it. If $S\{R\}$, then R is a *substructure* of S . We shall tend to shorten $S\{[R \wp U]\}$ as $S[R \wp U]$ when $[R \wp U]$ fills the hole $\{ \}$ of $S\{ \}$ exactly.

Congruence \approx on structures. Structures are partitioned by the smallest congruence \approx we obtain as reflexive, symmetric, transitive and contextual closure of the relation \sim whose defining clauses are (4), through (20) here below.

Negation		Associativity	
$\overline{\overline{\circ}} \sim \circ$	(4)	$(R \otimes (T \otimes V)) \sim ((R \otimes T) \otimes V)$	(12)
$\overline{\overline{R}} \sim R$	(5)	$\langle R \circ \langle T \circ V \rangle \rangle \sim \langle \langle R \circ T \rangle \circ V \rangle$	(13)
$\overline{[R \wp T]} \sim (\overline{R} \otimes \overline{T})$	(6)	$[R \wp [T \wp V]] \sim [[R \wp T] \wp V]$	(14)
$\overline{(R \otimes T)} \sim [\overline{R} \wp \overline{T}]$	(7)	Unit	
$\overline{\langle R \circ T \rangle} \sim \langle \overline{R} \circ \overline{T} \rangle$	(8)	$(\circ \otimes R) \sim R$	(15)
$\overline{[R]_a} \sim [\overline{R}]_a$	(9)	$\langle \circ \circ R \rangle \sim \langle R \circ \circ \rangle \sim R$	(16)
Symmetry		$[\circ \wp R] \sim R$	(17)
$[R \wp T] \sim [T \wp R]$	(10)	α -rule	
$(R \otimes T) \sim (T \otimes R)$	(11)	$[R]_a \sim R$ if $a \notin \text{fn}(R)$	(18)
		$[R^{a/b}]_a \sim [R]_b$ if $a \notin \text{fn}(R)$	(19)
		$[[R]_b]_a \sim [[R]_a]_b$	(20)

Contextual closure means that $S\{R\} \approx S\{T\}$ whenever $R \approx T$. We remark that Sdq is self-dual like Seq is. When introducing the logical rules we shall clarify why. Thanks to (20), we abbreviate $[\dots [R]_{a_1} \dots]_{a_n}$ as $[R]_{\vec{a}}$, where we may also interpret \vec{a} as one of the permutations of a_1, \dots, a_n .

The system SBVQ. It contains the set of inference rules in (21) here below. Every rule has form $\rho \frac{T}{R}$, *name* ρ , *premise* T , and *conclusion* R .

$\text{ai}\downarrow \frac{\circ}{[a \wp \overline{a}]}$	$\text{ai}\uparrow \frac{(a \otimes \overline{a})}{\circ}$	
$\text{q}\downarrow \frac{\langle [R \wp U] \circ [T \wp V] \rangle}{[\langle R \circ T \rangle \wp \langle U \circ V \rangle]}$	$\text{s} \frac{([R \wp T] \otimes U)}{[(R \otimes U) \wp T]}$	$\text{q}\uparrow \frac{\langle \langle R \circ T \rangle \otimes \langle U \circ V \rangle \rangle}{\langle (R \otimes U) \circ (T \otimes V) \rangle}$
$\text{u}\downarrow \frac{[[R \wp U]]_a}{[[R]_a \wp [U]_a]}$	$\text{u}\uparrow \frac{([R]_a \otimes [U]_a)}{[(R \otimes U)]_a}$	(21)

Every (instance of) inference rule can be used in any context, namely as $\rho \frac{S\{T\}}{S\{R\}}$ for any $S\{ \}$.

This means that, if a structure U matches R in $S\{ \}$, it can be rewritten to $S\{T\}$. This justifies calling R the *redex* of ρ , and T its *reduct*.

Up and down fragments of SBVQ. The set $\{\text{ai}\downarrow, \text{s}, \text{q}\downarrow, \text{u}\downarrow\}$ is the *down fragment* BVQ of SBVQ. The *up fragment* is $\{\text{ai}\uparrow, \text{s}, \text{q}\uparrow, \text{u}\uparrow\}$. So s belongs to both. The rule $\text{ai}\uparrow$ plays the role of the cut rule of sequent calculus. The down rule for Sdq restricts the following one [14]:

$$\frac{\forall a. [R \wp U]}{[\forall a. R \wp \exists a. U]}$$

to binding variable names only. Limiting Sdq to abstract variables implies that the difference between existentially, and universally quantified names disappears. The reason is that the cut-elimination will have no need to differentiate between the substitution of an existentially quantified variable for a universally quantified one, or vice versa. So, Sdq becomes self-dual.

Derivations vs. proofs. A *derivation* in SBVQ is either a structure or an instance of the above rules or a sequence of two derivations. Both \mathcal{D} , and \mathcal{E} will range over derivations. The topmost structure in a derivation is its *premise*. The bottommost is its *conclusion*. The *length* $|\mathcal{D}|$ of a derivation \mathcal{D} is the number of rule instances in \mathcal{D} . A derivation \mathcal{D} of a

structure R in SBVQ from a structure T in SBVQ, only using a subset $\mathbf{B} \subseteq \text{SBVQ}$ is $\mathcal{D} \Vdash_{\mathbf{B}} \frac{T}{R}$.

The equivalent *space-saving* form we shall tend to use is $\mathcal{D} : T \vdash_{\mathbf{B}} R$. The derivation $\mathcal{D} \Vdash_{\mathbf{B}} \frac{T}{R}$ is a *proof* whenever $T \approx \circ$. We denote it as $\mathcal{P} \Vdash_{\mathbf{B}} \frac{T}{R}$, or $\mathcal{P} : \vdash_{\mathbf{B}} R$. Both \mathcal{P} , and \mathcal{Q} will range

over proofs. In general, we shall drop \mathbf{B} when clear from the context. In a derivation, we write $\frac{T}{R} \xrightarrow{\rho_1, \dots, \rho_m, n_1, \dots, n_p}$, whenever we use the rules ρ_1, \dots, ρ_m to derive R from T with the help

of n_1, \dots, n_p instances of (4), \dots , (11). To avoid cluttering derivations, whenever possible, we shall tend to omit the use of negation axioms (4), \dots , (9), associativity axioms (12), (13), (14), and symmetry axioms (10), (11). This means we avoid writing all brackets, as in $[R \wp [T \wp U]]$, in favor of $[R \wp T \wp U]$, for example. Finally if, for example, $q > 1$ instances of some axiom (n) of (4), \dots , (20) occurs among n_1, \dots, n_p , then we write $(n)^q$.

Admissible and derivable rules. A rule ρ is *admissible* for the system SBVQ if $\rho \notin \text{SBVQ}$ and, for every derivation \mathcal{D} such that $\mathcal{D} : T \vdash_{\{\rho\} \cup \text{SBVQ}} R$, there is a derivation \mathcal{D}' such that

$\mathcal{D}' : T \vdash_{\text{SBVQ}} R$. A rule ρ is *derivable* in $\mathbf{B} \subseteq \text{SBVQ}$ if $\rho \notin \mathbf{B}$ and, for every instance $\rho \frac{T}{R}$, there exists a derivation \mathcal{D} in \mathbf{B} such that $\mathcal{D} : T \vdash_{\mathbf{B}} R$.

The rules in (22) recall a core set of rules derivable in SBV, hence in SBVQ.

$\text{id} \frac{\circ}{[R \wp \bar{R}]}$	$\text{mixp} \frac{(R \otimes T)}{\langle R \wp T \rangle}$	(22)
$\text{id} \frac{(R \otimes \bar{R})}{\circ}$	$\text{pmix} \frac{\langle R \wp T \rangle}{[R \wp T]}$	

General interaction down and up. In (22), *general interaction up* is id^\uparrow , derivable in the set $\{\text{id}^\uparrow, \text{s}, \text{q}^\uparrow, \text{u}^\uparrow\}$, reasoning by induction on $|R|$, and proceeding by cases on the form of R . We show the few steps of the proof, relative the case Sdq :

$$\begin{array}{c}
\frac{(9) \frac{([R]_a \otimes [\bar{R}]_a)}{([R]_a \otimes [\bar{R}]_a)}}{\text{id}^\uparrow \frac{([R]_a \otimes [\bar{R}]_a)}{[(R \otimes \bar{R})]_a}} \text{ ind. hypothesis} \\
\frac{\text{id}^\uparrow \frac{([R]_a \otimes [\bar{R}]_a)}{[(R \otimes \bar{R})]_a}}{(18) \frac{[\circ]_a}{\circ}}
\end{array}$$

Similar arguments apply to the cases relative to Not, CoPar, Seq, and Par. Symmetrically, *general interaction down* $i\downarrow$ is derivable in $\{ai\downarrow, s, q\downarrow, u\downarrow\}$.

General Seq-transitive up, and down rules. In (22) $t\downarrow$ is derivable by reasoning inductively on the size of $S\{\}$, and proceeding by cases on its structure, under the proviso (*) which says that $(\{a\} \cup \text{fn}(T)) \cap \text{bn}(S\{\}) = \emptyset$. If $S\{\} \approx \{\}$, then $t\downarrow$ is:

$$\begin{array}{c} \frac{\langle R \circ T \rangle}{\text{ai}\downarrow, (17), (16)} \frac{\langle R \circ \langle [a \circ \overline{a}] \circ [o \circ T] \rangle \rangle}{\text{q}\downarrow} \frac{\langle R \circ \langle [a \circ o] \circ \langle \overline{a} \circ T \rangle \rangle \rangle}{(17), (16)} \frac{\langle [R \circ o] \circ [a \circ \langle \overline{a} \circ T \rangle] \rangle}{\text{q}\downarrow} \frac{[\langle R \circ a \rangle \circ \langle o \circ \langle \overline{a} \circ T \rangle \rangle]}{(16)} \frac{[\langle R \circ a \rangle \circ \langle \overline{a} \circ T \rangle]}{\text{q}\downarrow} \end{array}$$

If $S\{\} \approx (S'\{\} \otimes U)$, then:

$$\frac{\frac{([S'\langle R \circ T \rangle] \otimes U)}{([S'\langle R \circ \overline{a} \rangle \circ \langle a \circ T \rangle] \otimes U)} \text{ind. hypothesis}}{([S'\langle R \circ \overline{a} \rangle \otimes U] \circ \langle a \circ T \rangle)} \text{s}$$

If $S\{\} \approx [S'\{\}]_p$, then:

$$\frac{\frac{[S'\langle R \circ T \rangle]_p}{[S'\langle R \circ \overline{a} \rangle \circ \langle a \circ T \rangle]_p} \text{ind. hypothesis}}{[S'\langle R \circ \overline{a} \rangle]_p \circ \langle a \circ T \rangle} \text{(18), u}\downarrow$$

The case with $S\{\} \approx [S'\{\} \circ U]$ is simpler than the two here above.

Mix rules. In (22) both mixp, and pmix, show a hierarchy between connectives: Par is the lowermost, Seq lies in the middle, and CoPar on top [5]. *Postfix mix rule* mixp is derivable in $\{q\uparrow\}$.

Finally, some properties that formalize simple derivations we can always build inside BVQ. The first one says when two structures R , and T of BVQ can be moved inside a context so that they get one aside the other.

Proposition 2.2 (Context extrusion) $S[R \circ T] \vdash_{\{q\downarrow, u\downarrow, s\}} [S\{R\} \circ T]$, for every S, R, T .

Proof By induction on $|S\{\}|$, proceeding by cases on the form of $S\{\}$. (Details in Appendix A).

The following statement highlights the scoping nature of Sdq. For proving it, it is enough to inspect the behavior of the rules in BVQ.

Fact 2.3 (Sdq is a scoping operator) Let a, U , and V be given.

1. If $\mathcal{D} : V \vdash_{\text{BVQ}} [U]_a$, then there exist R , and \mathcal{D}' such that $\mathcal{D} : [R]_a \vdash_{\text{BVQ}} [U]_a$.
2. For every R , if $\mathcal{D} : [R]_a \vdash_{\text{BVQ}} [U]_a$, then $\mathcal{D}' : R \vdash_{\text{BVQ}} U$, for some \mathcal{D}' .

The last property says that no new variable can be introduced in the course of a derivation.

Proposition 2.4 (BVQ is affine) In every $\mathcal{D} : T \vdash_{\text{BVQ}} R$, we have $|R| \geq |T|$.

Proof By induction on $|\mathcal{D}|$, proceeding by cases on its last rule ρ .

3 Splitting for SBVQ

We recall, and clean the proof of Splitting for SBVQ in [11, 12]. Splitting can be viewed as a generalization of cut-elimination for sequent calculus-like systems. Proving Splitting of SBVQ amounts to proving that SBVQ, and BVQ are equivalent, namely that every up-rule is admissible in BVQ, or, equivalently, that we can eliminate every up-rule from any derivation of SBVQ. Since $\text{a}\uparrow$ is an up-rule, and it plays the role of the cut rule, proving Splitting means proving also cut-elimination for SBVQ.

The first part of this section traces how Splitting, and some other properties it relies on, works to eliminate $\text{u}\uparrow$. The second part, Subsection 3.1, is for technical eyes interested to the full formal details.

Let us see how Splitting eliminates an occurrence $(*)$ of $\text{u}\uparrow$ from a proof \mathcal{P} of SBVQ, so focusing on the case that differentiates the proof of Splitting for SBVQ from the one for SBV. Let:

$$\text{u}\uparrow \frac{\mathcal{P}' \parallel S([R]_a \otimes [T]_a)}{S[(R \otimes T)]_a} (*)$$

be \mathcal{P} with $(*)$ the instance of $\text{u}\uparrow$ we want to eliminate. We are going to rewrite \mathcal{P} to a proof of BVQ with the same conclusion as \mathcal{P} , but without $(*)$. The first step to get rid of $(*)$ is Splitting (Theorem 3.5). The instance of Splitting we need, up to some details we can ignore at this level, is:

$$\text{If } S[(R \otimes T)]_a, \text{ then } \exists K, \vec{b} \text{ such that } \forall V, \text{ both } \frac{[[V \wp K]]_{\vec{b}}}{S\{V\}} \text{ and } \frac{\mathcal{Q}' \parallel}{[(R \otimes T) \wp K]}$$

We remark that extracting K , hidden inside \mathcal{Q} , might require many instances of Sdq to emerge, as the outermost occurrence $[\cdot]_{\vec{b}}$ in the premise of \mathcal{Q} shows. We can apply Splitting by taking \mathcal{P} — beware, not \mathcal{P}' — as \mathcal{Q} . Since V in \mathcal{Q} can be any, we choose $V \approx [(R \otimes T)]_a$, the conclusion of the instance of $\text{u}\uparrow$ we want to eliminate. From such an instance of \mathcal{Q} we get:

$$\frac{[[[(R \otimes T) \wp K]]_{\vec{b}}]}{\mathcal{Q}' \parallel S[(R \otimes T)]_a}$$

Now we extract from K the, usually called, killers of R , and T inside $(R \otimes T)$. Namely, we apply the following instance of Shallow splitting (Proposition 3.2) to the above \mathcal{Q}' :

$$\text{If } \frac{\mathcal{Q}'' \parallel}{[(R \otimes T) \wp K]}, \text{ then } \exists K_1, K_2, \vec{c} \text{ such that } \frac{[[K_1 \wp K_2]]_{\vec{c}}}{\mathcal{E} \parallel K} \text{ and } \frac{\mathcal{E}_1 \parallel}{[R \wp K_1]} \text{ and } \frac{\mathcal{E}_2 \parallel}{[T \wp K_2]}$$

which, once more, may let instances of Sdq to emerge. Composing \mathcal{Q}' , \mathcal{E} , \mathcal{E}_1 , and \mathcal{E}_2 , we get

the $(*)$ -free proof we are looking for:

$$\begin{array}{c}
\llbracket \circ \rrbracket_{c,b} \approx \circ \\
\mathcal{E}_2 \parallel \\
\llbracket [T \wp K_2] \rrbracket_{c,b} \\
\mathcal{E}_1 \parallel \\
\frac{\llbracket [([R \wp K_1] \otimes T) \wp K_2] \rrbracket_{c,b}}{\llbracket [(R \otimes T) \wp K_1 \wp K_2] \rrbracket_{c,b}} \text{ s} \\
\text{Proposition 2.2} \parallel \text{BVQ} \\
\llbracket [(R \otimes T) \wp \llbracket [K_1 \wp K_2] \rrbracket_{c'} \rrbracket_{c'} \rrbracket_b \\
\mathcal{E} \parallel \\
\llbracket [(R \otimes T) \wp K] \rrbracket_b \\
\mathcal{P}' \parallel \\
S \llbracket (R \otimes T) \rrbracket_a
\end{array}$$

It is a proof with the same conclusion as \mathcal{P} , without $(*)$, but with, at least, a couple of new instances of both $\text{u}\downarrow$, and s , the first one being “inside” Proposition 2.2

3.1 Details on Splitting

Proposition 3.1 (Provability of structures in BVQ) Let R , and T be structures, and a be a name, and \mathcal{P} , \mathcal{P}_1 , and \mathcal{P}_2 be proofs of BVQ.

1. $\mathcal{P} : \vdash_{\text{BVQ}} \langle R \wp T \rangle$ iff $\mathcal{P}_1 : \vdash_{\text{BVQ}} R$ and $\mathcal{P}_2 : \vdash_{\text{BVQ}} T$.
2. $\mathcal{P} : \vdash_{\text{BVQ}} (R \otimes T)$ iff $\mathcal{P}_1 : \vdash_{\text{BVQ}} R$ and $\mathcal{P}_2 : \vdash_{\text{BVQ}} T$.
3. $\mathcal{P} : \vdash_{\text{BVQ}} \llbracket R \rrbracket_a$ iff $\mathcal{P}' : \vdash_{\text{BVQ}} R\{^b/_a\}$, for every variable b .

Proof “If implication”. The proofs of 1 and 2, given in [5] by induction on $|\mathcal{P}|$ inside BV, extend to the cases when the last rule of \mathcal{P} is $\text{u}\downarrow$. Indeed, the redex of $\text{u}\downarrow$ can only be inside R or T . Concerning 3, the assumption implies the existence of $\mathcal{P}' : \vdash R\{^a/_a\}$, namely of $\mathcal{P}' : \vdash R$. So, we can “wrap” \mathcal{P}' with $\llbracket \cdot \rrbracket_a$, exploiting (18), and apply every rule of \mathcal{P}' deep in the proof \mathcal{P} we are building.

“Only if implication”. In all the three cases, the proof is by induction on $|\mathcal{P}|$, proceeding by cases on its last rule ρ . Concerning points 1, and 2 a redex can only be inside R or T . So, the application of the inductive hypothesis is immediate. Instead, a may not belong to $\text{fn}(R)$ in Point 3. If this is true, then (18) implies that every instance of \mathcal{P}' with b in place of a exists. The reason is that, by definition, the substitution (3) distributes over structures, preserving the scope of every instance of Sdq . Otherwise, if $a \in \text{fn}(R)$, then the redex of ρ can only be inside R . So, we can conclude thanks to the inductive hypothesis.

Proposition 3.2 (Shallow Splitting in BVQ) Let R, T , and P be structures, and a be a name, and \mathcal{P} be a proof of BVQ.

1. If $\mathcal{P} : \vdash_{\text{BVQ}} [\langle R \wp T \rangle \wp P]$, then there are $\mathcal{D} : \langle P_1 \wp P_2 \rangle \vdash_{\text{BVQ}} P$, and $\mathcal{P}_1 : \vdash_{\text{BVQ}} [R \wp P_1]$, and $\mathcal{P}_2 : \vdash_{\text{BVQ}} [T \wp P_2]$, for some P_1 , and P_2 .
2. If $\mathcal{P} : \vdash_{\text{BVQ}} [(R \otimes T) \wp P]$, then there are $\mathcal{D} : [P_1 \wp P_2] \vdash_{\text{BVQ}} P$, and $\mathcal{P}_1 : \vdash_{\text{BVQ}} [R \wp P_1]$, and $\mathcal{P}_2 : \vdash_{\text{BVQ}} [T \wp P_2]$, for some P_1 , and P_2 .
3. Let $\mathcal{P} : \vdash_{\text{BVQ}} [R \wp P]$ with $R \approx [l_1 \wp \dots \wp l_m]$, such that $i \neq j$ implies $l_i \neq \overline{l_j}$, for every $i, j \in \{1, \dots, m\}$, and $m > 0$. Then, for every structure R_0 , and R_1 , if $R \approx [R_0 \wp R_1]$, there exists $\mathcal{D} : \overline{R_1} \vdash_{\text{BVQ}} [R_0 \wp P]$.

4. If $\mathcal{P} : \vdash \llbracket R \rrbracket_a \wp P$, then there are $\mathcal{D} : \llbracket T \rrbracket_a \vdash_{\text{BVQ}} P$, and $\mathcal{P}' : \vdash_{\text{BVQ}} [R \wp T]$, for some T .

Proof Following [5], both statements 1, and 2 must be proved simultaneously. We reason by induction on the lexicographic order of the pair $(|V|, |\mathcal{P}|)$, where V is one between $\llbracket R \wp T \rrbracket$ or $\llbracket (R \otimes T) \rrbracket$, proceeding by cases on the last rule ρ of \mathcal{P} .

Point 3 relies on points 1, 2. It holds by induction on $(|R|, |\mathcal{P}|)$, proceeding by cases on the last rule of \mathcal{P} . Point 4 relies on points 1, 2. It holds by induction on $(\llbracket R \rrbracket_a, |\mathcal{P}|)$, proceeding by cases on the last rule of \mathcal{P} . (Details in Appendix B).

Remark 3.3 The proviso “ $i \neq j$ implies $l_i \neq \bar{l}_j$, for every $i, j \in \{1, \dots, m\}$ ” of Point (3) in Proposition 3.2 serves to let the killer of every l be inside $[R_0 \wp P]$.

Proposition 3.4 here below says that $S\{ \}$ supplies the “context” U , required for proving R , no matter which structure fills the hole of $S\{ \}$.

Proposition 3.4 (Context Reduction in BVQ) Let R be a structure, and $S\{ \}$ be a context such that $\mathcal{P} : \vdash_{\text{BVQ}} S\{R\}$. There are a structure U , and, possibly, some variables \vec{b} such that, for every V , if $\text{fn}(V) \cap \text{bn}(R) = \emptyset$, then both $\mathcal{D} : \llbracket [V \wp U] \rrbracket_{\vec{b}} \vdash_{\text{BVQ}} S\{V\}$, and $\mathcal{Q} : \vdash_{\text{BVQ}} [R \wp U]$.

Proof The proof is by induction on $|S\{ \}|$, proceeding by cases on the form of $S\{ \}$. (Details in Appendix C).

Theorem 3.5 (Splitting in BVQ) Let R , and T , be structures, and $S\{ \}$ be a context.

1. If $\mathcal{P} : \vdash_{\text{BVQ}} S\langle R \wp T \rangle$, then there are structures K_1, K_2 , and, possibly, some variables \vec{b} such that, for every V with $\text{fn}(V) \cap \text{bn}(\langle R \wp T \rangle) = \emptyset$, there are $\mathcal{D} : \llbracket [V \wp \langle K_1 \wp K_2 \rangle] \rrbracket_{\vec{b}} \vdash_{\text{BVQ}} S\{V\}$, and $\mathcal{P}_1 : \vdash_{\text{BVQ}} [R \wp K_1]$, and $\mathcal{P}_2 : \vdash_{\text{BVQ}} [T \wp K_2]$.
2. If $\mathcal{P} : \vdash_{\text{BVQ}} S(R \otimes T)$, then there are structures K_1, K_2 , and, possibly, some variables \vec{b} such that, for every V with $\text{fn}(V) \cap \text{bn}(R \otimes T) = \emptyset$, there are $\mathcal{D} : \llbracket [V \wp K_1 \wp K_2] \rrbracket_{\vec{b}} \vdash_{\text{BVQ}} S\{V\}$, and $\mathcal{P}_1 : \vdash_{\text{BVQ}} [R \wp K_1]$, and $\mathcal{P}_2 : \vdash_{\text{BVQ}} [T \wp K_2]$.
3. If $\mathcal{P} : \vdash_{\text{BVQ}} S\llbracket R \rrbracket_a$, then there are a structure K , and, possibly, some variables \vec{b} such that, for every V with $\text{fn}(V) \cap \text{bn}(\llbracket R \rrbracket_a) = \emptyset$, there exist $\mathcal{D} : \llbracket [V \wp K] \rrbracket_{\vec{b}} \vdash_{\text{BVQ}} S\{V\}$, and $\mathcal{P}' : \vdash_{\text{BVQ}} [R \wp K]$.

Proof We obtain the proof of the three statements by composing Context Reduction (Proposition 3.4), and Shallow Splitting (Proposition 3.2) in this order. (Details in Appendix D).

Theorem 3.6 (Admissibility of the up fragment for BVQ) The set $\{\text{ai}\uparrow, \text{q}\uparrow, \text{u}\uparrow\}$ in SBVQ is admissible for BVQ.

Proof Use Splitting (Theorem 3.5), and Shallow Splitting (Proposition 3.2) (Details in Appendix E.)

4 Linear λ -calculus mapped to BVQ

To show that Seq is not an extemporaneous logical operator we interpret it as binder that, together with Seq , models the renaming mechanism of linear β -reduction.

Linear λ -calculus. We recall that linear λ -calculus can be viewed as a pair (linear λ -terms, linear operational semantics). Let \mathcal{V} be a countable set of variable names we range over by x, y, w, z . We call \mathcal{V} the *set of λ -variables*. The set of *linear λ -terms* is $\Lambda = \bigcup_{X \subset \mathcal{V}} \Lambda_X$ we range over by M, N, P, Q . For every $X \subset \mathcal{V}$, the set Λ_X contains the *linear λ -terms whose free variables are in X* , and which we define as follows: (i) $x \in \Lambda_{\{x\}}$; (ii) $\lambda x.M \in \Lambda_X$ if $M \in \Lambda_{X \cup \{x\}}$; (iii) $(M)N \in \Lambda_{X \cup Y}$ if $M \in \Lambda_X$, $N \in \Lambda_Y$, and $X \cap Y = \emptyset$; (iv) $M \{^P/_x\} \in \Lambda_{X \cup Y}$ if $M \in \Lambda_{X \cup \{x\}}$, $P \in \Lambda_Y$, and $X \cap Y = \emptyset$. The linear operational semantics that rewrites linear λ -terms is the relation $\Rightarrow \subseteq \Lambda \times \Lambda$ here below:

$$\begin{array}{c}
\text{refl} \frac{}{M \Rightarrow M} \quad \beta \frac{}{(\lambda x.M)N \Rightarrow M\{^N/_x\}} \quad \text{tra} \frac{M \Rightarrow P \quad P \Rightarrow N}{M \Rightarrow N} \\
\text{f} \frac{M \Rightarrow N}{\lambda x.M \Rightarrow \lambda x.N} \quad @l \frac{M \Rightarrow N}{(M)P \Rightarrow (N)P} \quad @r \frac{M \Rightarrow N}{(P)M \Rightarrow (P)N}
\end{array} \quad (23)$$

where $M\{^N/_x\}$ is the usual clash-free *substitution*, that replaces N for the forcefully single occurrence of x in M . We remark that (23) is the reflexive, contextual, and transitive closure of linear β -reduction we find in rule β . Finally, $|M \Rightarrow N|$ denotes the *number of instances of rules* in (23), used to derive some given $M \Rightarrow N$.

The map $\llbracket \cdot \rrbracket_o$. We define it here below, to map terms of Λ into structures of BVQ.

$$\begin{array}{ll}
\llbracket x \rrbracket_o = \langle x \circ \bar{o} \rangle \text{ with } \bar{o} \text{ fresh} & (24a) \\
\llbracket \lambda x.M \rrbracket_o = \llbracket M \rrbracket_{o \downarrow x} & (24b) \\
\llbracket (M)N \rrbracket_o = \llbracket [\llbracket M \rrbracket_p \circ \llbracket N \rrbracket_q] \circ \langle p \circ \bar{o} \rangle \rrbracket_p & (24c)
\end{array}$$

For every linear λ -term M , the structure $\llbracket M \rrbracket_o$ is such that (i) o is a unique output channel, and (ii) every free variable of M is used as positive atom name that plays the role of input channel. Clause (24a) associates the input channel x to the fresh output channel o . Intuitively, x shall be eventually *forwarded* to o , in accordance with terminology taken from [9]. Clause (24b) uses Sdq to abstract on the input channel x . This means to let x ready to merge with any output channel of a linear λ -term that has to be substituted for x . Such a channel comes from the argument of an application, as translated by (24c). It wraps $\llbracket N \rrbracket_q$, abstracting on its output channel q thanks to Sdq . So, thanks to Sdq , linear β -reduction, and its substitution mechanism, become an identification of channel names inside BVQ, as follows:

$$\begin{array}{c}
(18) \frac{\llbracket M\{^N/_x\} \rrbracket_o}{\llbracket M\{^N/_x\} \rrbracket_{o \downarrow p}} \\
\text{mt} \downarrow \frac{\llbracket M\{^N/_x\} \rrbracket_{o \downarrow p}}{\llbracket [\llbracket M\{^N/_x\} \rrbracket_p \circ \langle p \circ \bar{o} \rangle] \rrbracket_p} \\
(18) \frac{\llbracket [\llbracket M\{^N/_x\} \rrbracket_p \circ \langle p \circ \bar{o} \rangle] \rrbracket_p}{\llbracket [\llbracket M \rrbracket_p \circ \llbracket N \rrbracket_x] \circ \langle p \circ \bar{o} \rangle \rrbracket_p} \\
\text{subst} \frac{\llbracket [\llbracket M \rrbracket_p \circ \llbracket N \rrbracket_x] \circ \langle p \circ \bar{o} \rangle \rrbracket_p}{\llbracket [\llbracket M \rrbracket_p \rrbracket_x \circ \llbracket N \rrbracket_x] \circ \langle p \circ \bar{o} \rangle \rrbracket_p} \\
u \downarrow \frac{\llbracket [\llbracket M \rrbracket_p \rrbracket_x \circ \llbracket N \rrbracket_x] \circ \langle p \circ \bar{o} \rangle \rrbracket_p}{\llbracket [\llbracket M \rrbracket_p \rrbracket_x \circ \llbracket N \rrbracket_q] \circ \langle p \circ \bar{o} \rangle \rrbracket_p} \\
(19) \frac{\llbracket [\llbracket M \rrbracket_p \rrbracket_x \circ \llbracket N \rrbracket_q] \circ \langle p \circ \bar{o} \rangle \rrbracket_p}{\llbracket (\lambda x.M)N \rrbracket_o \equiv \llbracket [\llbracket M \rrbracket_p \rrbracket_x \circ \llbracket N \rrbracket_q] \circ \langle p \circ \bar{o} \rangle \rrbracket_p}
\end{array} \quad (25)$$

In (25) here above (i) (19) holds because we have that $\llbracket N \rrbracket_q \downarrow q \approx \llbracket N \rrbracket_q \{^x/_q\} \downarrow x \approx \llbracket N \rrbracket_x \downarrow x$ holds thanks to the uniqueness of input, and output channels, and thanks to Seq which never

confuses left, and right-hand sides of $\langle R \prec T \rangle$, (ii) the instance of $u\downarrow$ identifies the input channel x of $\llbracket M \rrbracket_o \downarrow_x$ with the output channel x of $\llbracket N \rrbracket_x$, after its renaming by means of (19), (iii) we are going to show that both subst , and $\text{mt}\downarrow$ are derivable in BVQ, with the second one being a specialization of the transitivity $\text{t}\downarrow$, and (v) the two occurrences of (18) apply because x and p disappear.

5 Completeness of BVQ w.r.t. Linear λ -calculus

Completeness says that we can mimic every computation step of linear λ -calculus as proof-reconstruction inside BVQ.

Theorem 5.1 (Completeness of BVQ) For every M , and N , and o , if $M \Rightarrow N$, then $\mathcal{D} : \llbracket N \rrbracket_o \vdash_{\text{BVQ}} \llbracket M \rrbracket_o$.

The proof relies on some technical lemma that we detail out in the coming lines.

Lemma 5.2 (Output names are linear) For every M , and o , the output name o of $\llbracket M \rrbracket_o$ occurs once.

Proof By induction on the definition of $\llbracket \cdot \rrbracket_o$, proceeding by cases on the form of M .

Lemma 5.3 (Substitution in BVQ) For every M, N, p, o , and x , such that $x \in \text{fn}(\llbracket M \rrbracket_o)$, in BVQ, we can derive:

$$\text{mt}\downarrow \frac{\llbracket M \rrbracket_o}{[\llbracket M \rrbracket_p \wp \langle p \prec \overline{o} \rangle]} \quad \text{subst} \frac{\llbracket M\{^N/x\} \rrbracket_o}{[\llbracket M \rrbracket_o \wp \llbracket N \rrbracket_x]}$$

Proof Concerning $\text{mt}\downarrow$, we reason inductively on the size of $\llbracket \cdot \rrbracket_o$, proceeding by cases on M . (Details in Appendix F.) Concerning subst , we reason inductively on the size of $[\llbracket M \rrbracket_o \wp \llbracket N \rrbracket_x]$, exploiting $\text{mt}\downarrow$. (Details in Appendix G.)

Lemma 5.4 (Linear β reduction in BVQ) For every M, N, o , and x , in BVQ, we can derive:

$$\text{beta} \frac{\llbracket M\{^N/x\} \rrbracket_o}{\llbracket (\lambda x. M) N \rrbracket_o}$$

Proof The rule beta is derived in (25) exploiting the definition of $\llbracket \cdot \rrbracket_o$, and Lemma 5.3.

Proof of Theorem 5.1. By induction on $|M \Rightarrow N|$, proceeding by cases on the last rule in (23) used for proving $M \Rightarrow N$. If the last rule is beta , then Lemma 5.4 implies the thesis. Let the last rule be tra . The inductive hypothesis implies the existence of \mathcal{D}_0 , and \mathcal{D}_1 :

$$\begin{array}{c} \llbracket N \rrbracket_o \\ \mathcal{D}_1 \parallel \\ \llbracket P \rrbracket_o \\ \mathcal{D}_0 \parallel \\ \llbracket M \rrbracket_o \end{array}$$

In all the remaining cases we proceed as here above, exploiting that BVQ is a DI system, so we can apply deeply, namely in any context, every of its rules.

Remark 5.5 As a corollary, under the same assumption as Theorem 5.1, we have $\vdash_{\text{BVQ}} [\llbracket M \rrbracket_o \wp \llbracket N \rrbracket_o]$ because we can derive $i\downarrow$ in BVQ, and we can plug it on top of \mathcal{D} .

6 Conclusions and future work

On the computational interpretation side of proof-search inside BVQ, this work makes no reference to soundness of BVQ w.r.t. linear λ -calculus. Soundness is the reverse of completeness.

For every M, N , and o , if $\mathcal{D} \Vdash_{\text{BVQ}} M \Rightarrow N$, then $M \Rightarrow N$. A counter example to it is:

$$\begin{aligned} \llbracket N \rrbracket_o & \\ \llbracket M \rrbracket_o & \\ ((\lambda x. M) P) Q \rrbracket_o &= \llbracket [\llbracket M \rrbracket_{s \downarrow x} \otimes \llbracket P \rrbracket_{p \downarrow p} \otimes \langle s \ast \bar{r} \rangle]_s \otimes \llbracket Q \rrbracket_{q \downarrow q} \otimes \langle r \ast \bar{o} \rangle]_r \\ &\approx \llbracket [\llbracket M \rrbracket_{s \downarrow x} \otimes \llbracket P \rrbracket_{p \downarrow p} \otimes \langle s \ast \bar{r} \rangle]_s \otimes \llbracket [\llbracket Q \rrbracket_{q \downarrow q} \otimes \langle r \ast \bar{o} \rangle]_r \\ &\approx \llbracket [\llbracket M \rrbracket_{s \downarrow x} \otimes \llbracket Q \rrbracket_{q \downarrow q} \otimes \llbracket P \rrbracket_{p \downarrow p} \otimes \langle s \ast \bar{r} \rangle]_s \otimes \langle r \ast \bar{o} \rangle]_r \end{aligned}$$

where we would erroneously substitute (the mapping of) Q for (the mapping of) x in (the mapping of) M . We think essentially two ways exist to react to the lack of soundness of BVQ w.r.t. linear λ -calculus. The first is in [11, 12] which proves a weak, and not so interesting form of soundness. The second way is replacing the target language linear λ -calculus, so moving towards the programme that [2] begins. It suggests that the natural computational paradigm w.r.t. which BVQ can be sound, is some extension of CC_{sp} , the fragment of Milner CCS with sequential and parallel composition only. This is coming work, indeed.

On the proof-theoretical side, whose concern is the minimal, and incremental extension of SBV, an example of which is SBVQ, we plan to keep investigating self-dual operators. By means of a self-dual operator, and in accordance with the proof-search-as-computation paradigm, we plan to model non deterministic choice. Candidate rules that model a self-dual non-deterministic choice are¹:

$$\text{p}\downarrow \frac{[[R \otimes T] \oplus [U \otimes T]]}{[[R \oplus U] \otimes T]} \quad \text{p}\uparrow \frac{[(R \oplus U) \otimes T]}{[(R \otimes T) \oplus (U \otimes T)]}$$

We think they are interesting because they would internalize the non deterministic choice that we apply at the meta-level when searching for proofs, or derivations, inside SBVQ or SBV.

References

- [1] Kai Brännler and Richard McKinley. An algorithmic interpretation of a deep inference system. In Iliano Cervesato, Helmut Veith, and Andrei Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning (LPAR)*, volume 5330 of *Lecture Notes in Computer Science*, pages 482–496. Springer-Verlag, 2008. <http://www.iam.unibe.ch/~kai/Papers/2008aidis.pdf>.
- [2] Paola Bruscoli. A purely logical account of sequentiality in proof search. In Peter J. Stuckey, editor, *Logic Programming, 18th International Conference*, volume 2401 of *Lecture Notes in Computer Science*, pages 302–316. Springer-Verlag, 2002. <http://cs.bath.ac.uk/pb/bv1/bv1.pdf>.
- [3] Jean-Yves Girard, Paul Taylor, and Yves Lafont. *Proofs and types*. Cambridge University Press, New York, NY, USA, 1989.
- [4] Nicolas Guenot. *Nested Deduction in Logical Foundations for Computation*. PhD thesis, Ecole Polytechnique — Laboratoire d’Informatique (LIX), rue de Saclay, 91128 Palaiseau cedex, 2013.

¹The conjecture about the existence of the two rules $\text{p}\downarrow$, and $\text{p}\uparrow$, that model non-deterministic choice, results from discussions with Alessio Guglielmi.

- [5] Alessio Guglielmi. A system of interaction and structure. *ACM Transactions on Computational Logic*, 8(1):1–64, 2007. <http://cs.bath.ac.uk/ag/p/SystIntStr.pdf>.
- [6] Alessio Guglielmi and Lutz Straßburger. Non-commutativity and MELL in the calculus of structures. In L. Fribourg, editor, *CSL 2001*, volume 2142 of *Lecture Notes in Computer Science*, pages 54–68. Springer-Verlag, 2001. <http://cs.bath.ac.uk/ag/p/NoncMELLCoS.pdf>.
- [7] Alessio Guglielmi and Lutz Straßburger. A non-commutative extension of MELL. In M. Baaz and A. Voronkov, editors, *LPAR 2002*, volume 2514 of *Lecture Notes in Computer Science*, pages 231–246. Springer-Verlag, 2002. <http://www.lix.polytechnique.fr/~lutz/papers/NEL.pdf>.
- [8] Alessio Guglielmi and Lutz Straßburger. A system of interaction and structure v: the exponentials and splitting. *Mathematical Structures in Computer Science*, 21(3):563–584, 2011.
- [9] K. Honda and N. Yoshida. On the Reduction-based Process Semantics. *Theoretical Computer Science*, (151):437–486, 1995.
- [10] Harry G. Mairson. Linear lambda calculus and ptime-completeness. *J. Funct. Program.*, 14(6):623–633, 2004.
- [11] Luca Roversi. Linear lambda calculus with explicit substitutions as proof-search in Deep Inference. <http://arxiv.org/abs/1011.3668>. November 2010.
- [12] Luca Roversi. Linear Lambda Calculus and Deep Inference. In Luke Ong, editor, *TLCA 2011 - 10th Typed Lambda Calculi and Applications, Part of RDP'11*, volume 6690 of *ARCoSS/LNCS*, pages 184 – 197. Springer, 2011.
- [13] Lutz Straßburger. System NEL is undecidable. In Ruy De Queiroz, Elaine Pimentel, and Lucília Figueiredo, editors, *10th Workshop on Logic, Language, Information and Computation (WoLLIC)*, volume 84 of *Electronic Notes in Theoretical Computer Science*. Elsevier, 2003. http://www.lix.polytechnique.fr/~lutz/papers/NELundec_wollic03.pdf.
- [14] Lutz Straßburger. Some Observations on the Proof Theory of Second Order Propositional Multiplicative Linear Logic. In Pierre-Louis Curien, editor, *Typed Lambda Calculi and Applications*, volume 5608 of *Lecture Notes in Computer Science*, pages 309–324. Springer-Verlag, 2009. <http://www.lix.polytechnique.fr/~lutz/papers/ObsPT-MLL2-finalforTLCA09.pdf>.
- [15] Lutz Straßburger and Alessio Guglielmi. A system of interaction and structure iv: The exponentials and decomposition. *ACM Trans. Comput. Log.*, 12(4):23, 2011.
- [16] Alwen Tiu. A system of interaction and structure II: The need for deep inference. *Logical Methods in Computer Science*, 2(2:4):1–24, 2006. <http://arxiv.org/pdf/cs.LO/0512036>.
- [17] Steffen van Bakel and Maria Grazia Vigliotti. A logical interpretation of the λ -calculus into the π -calculus, preserving spine reduction and types. In *CONCUR*, pages 84–98, 2009.

A Proof of *Context extrusion* (Proposition 2.2, page 7)

By induction on $|S\{ \ }|$, proceeding by cases on the form of $S\{ \ }$. The base is with $S\{ \ } \equiv \{ \ }$. The statement holds simply because (i) $S[R \wp T] \equiv [S\{R\} \wp T] \equiv [R \wp T]$, and (ii) $[R \wp T]$ is a structure, so, by definition, a derivation.

As a *first case*, let $S\{ \ } \equiv \langle S'\{ \ } \triangleleft U \rangle$. Then:

$$\frac{\begin{array}{c} \langle S'[R \wp T] \triangleleft U \rangle \equiv S[R \wp T] \\ \mathcal{D} \parallel \\ \langle [S\{R\} \wp T] \triangleleft U \rangle \end{array}}{\text{q}\downarrow, (16) \quad [S\{R\} \wp T] \equiv [\langle S'\{R\} \triangleleft U \rangle \wp T]}$$

where \mathcal{D} exists by inductive hypothesis which holds thanks to $|S'\{ \ }| < |S\{ \ }|$. If, instead $S\{ \ } \equiv (S'\{ \ } \otimes U)$, we can proceed as here above, using \otimes in place of \wp .

As a *second case*, let $S\{ \ } \equiv [S'\{ \ }]_a$. Without loss of generality, thanks to (19), we can assume $a \notin \text{fn}(T)$. Then:

$$\frac{\begin{array}{c} [S'[R \wp T]]_a \equiv S[R \wp T] \\ \mathcal{D} \parallel \\ [[S'\{R\} \wp T]]_a \end{array}}{\text{u}\downarrow, (18) \quad [S\{R\} \wp T] \equiv [[S'\{R\}]_a \wp T] \equiv [[S'\{R\}]_a \wp [T]_a]}$$

where \mathcal{D} exists by inductive hypothesis which holds thanks to $|S'\{ \ }| < |S\{ \ }|$.

B Proof of *Shallow Splitting* (Proposition 3.2, page 9)

Proof of Points 1 and 2. We prove the two statements simultaneously, by induction on the lexicographic order $(|U|, |\mathcal{P}|)$, where U is one among $\langle R \triangleleft T \rangle \wp P$, and $[(R \otimes T) \wp P]$, proceeding by cases on the last rule ρ of \mathcal{P} .

As a *first case* for both points 1 and 2 we assume the redex of ρ is inside one among R, T or P . So, \mathcal{P} is one between:

$$\frac{\begin{array}{c} \mathcal{P}' \parallel \\ [\langle R' \triangleleft T' \rangle \wp P'] \end{array}}{\rho \quad [\langle R \triangleleft T \rangle \wp P]} \quad \frac{\begin{array}{c} \mathcal{P}'' \parallel \\ [(R' \otimes T') \wp P'] \end{array}}{\rho \quad [(R \otimes T) \wp P]}$$

where only one among R', T', P' is the reduct of ρ . We can conclude by applying the inductive hypothesis on \mathcal{P}' , or \mathcal{P}'' , and ρ in the obvious way.

As a *second case* of Point 1 let ρ be $\text{q}\downarrow$ with $[\langle \langle R' \triangleleft R'' \rangle \triangleleft T \rangle \wp [\langle P' \triangleleft P'' \rangle \wp P''']]$ as its redex. So, \mathcal{P} can be:

$$\frac{\begin{array}{c} \mathcal{P}' \parallel \\ [\langle [R' \wp P'] \triangleleft [\langle R'' \triangleleft T \rangle \wp P''] \rangle \wp P'''] \end{array}}{\text{q}\downarrow \quad \frac{\begin{array}{c} [[\langle R' \triangleleft \langle R'' \triangleleft T \rangle \rangle \wp \langle P' \triangleleft P'' \rangle] \wp P'''] \\ (13), (14) \quad [[\langle R' \triangleleft R'' \rangle \triangleleft T] \wp [\langle P' \triangleleft P'' \rangle \wp P''']] \end{array}}{\mathcal{P}' \parallel}}$$

Thanks to $[[\langle \langle R' \triangleleft R'' \rangle \triangleleft T \rangle \wp [\langle P' \triangleleft P'' \rangle \wp P''']] = [[\langle [R' \wp P'] \triangleleft [\langle R'' \triangleleft T \rangle \wp P''] \rangle \wp P''']]$ and $|\mathcal{P}'| < |\mathcal{P}|$ the inductive hypothesis holds on \mathcal{P}' which implies $\mathcal{E} : \langle P_1 \triangleleft P_2 \rangle \vdash P'''$, and $\mathcal{P}'' : \vdash [[R' \wp P'] \wp P_1]$, and $\mathcal{Q} : \vdash [[\langle R'' \triangleleft T \rangle \wp P''] \wp P_2]$.

Thanks to $[[[\langle R'' \triangleleft T \rangle \wp P''] \wp P_2]] < [[\langle [R' \wp P'] \triangleleft [\langle R'' \triangleleft T \rangle \wp P''] \rangle \wp P''']]$ the inductive hypothesis holds on \mathcal{Q} which implies $\mathcal{E}' : \langle U_1 \triangleleft U_2 \rangle \vdash [P'' \wp P_2]$, and $\mathcal{Q}' : \vdash [R'' \wp U_1]$, and $\mathcal{Q}'' : \vdash [T \wp U_2]$.

The first derivation and the first proof of BVQ in the statement we have to prove are:

$$\begin{array}{c}
\frac{\langle \langle [P' \wp P_1] \wp U_1 \rangle \wp U_2 \rangle}{\langle \langle [P' \wp P_1] \wp U_1 \rangle \wp U_2 \rangle} \quad (13) \\
\mathcal{E}' \parallel \\
\frac{\langle \langle [P' \wp P_1] \wp [P'' \wp P_2] \rangle \rangle}{\langle \langle [P' \wp P_1] \wp [P'' \wp P_2] \rangle \rangle} \quad \text{ql} \\
\mathcal{E} \parallel \\
\langle \langle [P' \wp P_1] \wp [P'' \wp P_2] \rangle \rangle
\end{array}
\quad
\begin{array}{c}
\mathcal{P}'' \parallel \\
\frac{[[R' \wp P'] \wp P_1]}{\langle \langle [R' \wp P'] \wp P_1 \rangle \wp \circ \rangle} \quad (16) \\
\mathcal{Q}' \parallel \\
\frac{\langle \langle [R' \wp P'] \wp P_1 \rangle \wp [R'' \wp U_1] \rangle}{\langle \langle [R' \wp P'] \wp P_1 \rangle \wp [R'' \wp U_1] \rangle} \quad (14) \\
\text{ql} \\
\langle \langle [R' \wp P'] \wp P_1 \rangle \wp [R'' \wp U_1] \rangle
\end{array}$$

The second proof of BVQ in the statement we have to prove is \mathcal{Q}'' .

The situation with $\rho \equiv \text{ql}$ and $[\langle R \wp T \rangle \wp [\langle P' \wp P'' \rangle \wp P''']]$ its redex is analogous to one one just developed.

As a *third case* of Point 1 let ρ be ql with $[\langle R \wp T \rangle \wp [\langle P' \wp P'' \rangle \wp [U' \wp U'']]]$ as its redex. So, \mathcal{P} can be:

$$\begin{array}{c}
\mathcal{P}' \parallel \\
\frac{[\langle P' \wp [\langle R \wp T \rangle \wp P'' \rangle] \wp [U' \wp U'']]}{[[\langle \circ \wp \langle R \wp T \rangle \rangle \wp \langle P' \wp P'' \rangle] \wp [U' \wp U'']] \quad \text{ql}, (16)} \\
(14), (16) \quad \frac{[[\langle \circ \wp \langle R \wp T \rangle \rangle \wp \langle P' \wp P'' \rangle] \wp [U' \wp U'']]}{[\langle R \wp T \rangle \wp [\langle P' \wp P'' \rangle \wp [U' \wp U'']]]}
\end{array}$$

Thanks to $[[\langle R \wp T \rangle \wp [\langle P' \wp P'' \rangle \wp [U' \wp U'']]] = [[\langle P' \wp [\langle R \wp T \rangle \wp P'' \rangle] \wp [U' \wp U'']]]$ and $|\mathcal{P}'| < |\mathcal{P}|$ the inductive hypothesis holds on \mathcal{P}' yielding $\mathcal{E} : \langle P_1 \wp P_2 \rangle \vdash [U' \wp U'']$, and $\mathcal{P}'' : \vdash [P' \wp P_1]$, and $\mathcal{Q} : \vdash [[\langle R \wp T \rangle \wp P''] \wp P_2]$.

Thanks to $[[[\langle R \wp T \rangle \wp P''] \wp P_2]] < [[\langle R \wp T \rangle \wp [\langle P' \wp P'' \rangle \wp [U' \wp U'']]]]$ and $|\mathcal{P}'| < |\mathcal{P}|$ the inductive hypothesis holds on \mathcal{Q} yielding $\mathcal{E}' : \langle U_1 \wp U_2 \rangle \vdash [P' \wp P_2]$, and $\mathcal{Q}' : \vdash [R \wp U_1]$, and $\mathcal{Q}'' : \vdash [T \wp U_2]$.

Both \mathcal{Q}' , and \mathcal{Q}'' are the two proofs of BVQ of the statement we have to prove. The derivation of BVQ is:

$$\begin{array}{c}
\frac{\langle U_1 \wp U_2 \rangle}{\langle \circ \wp \langle U_1 \wp U_2 \rangle \rangle} \quad (16) \\
\mathcal{P}'' \parallel \\
\langle \langle [P' \wp P_1] \wp \langle U_1 \wp U_2 \rangle \rangle \rangle \\
\mathcal{E}' \parallel \\
\frac{\langle \langle [P' \wp P_1] \wp [P'' \wp P_2] \rangle \rangle}{\langle \langle [P' \wp P_1] \wp [P'' \wp P_2] \rangle \rangle} \quad \text{ql} \\
\mathcal{E} \parallel \\
\langle \langle [P' \wp P_1] \wp [P'' \wp P_2] \rangle \rangle
\end{array}$$

As a *fourth case* of Point 1 let ρ be s with $[\langle R \wp T \rangle \wp [(P' \wp P'') \wp P''']]$ as its redex. So, \mathcal{P} can be:

$$\begin{array}{c}
\mathcal{P}' \parallel \\
\frac{[[[\langle R \wp T \rangle \wp P'] \wp P''] \wp P''']] {[(P' \wp P'') \wp P''']] \quad (10)} \\
\text{s} \\
\frac{[[[(P' \wp P'') \wp P''] \wp \langle R \wp T \rangle] \wp P''']] {[(P' \wp P'') \wp P''']] \quad (14), (10)} \\
\langle \langle [P' \wp P_1] \wp [P'' \wp P_2] \rangle \rangle
\end{array}$$

Thanks to $[[[\langle R \wp T \rangle \wp P'] \wp P'']] = [[[(\langle R \wp T \rangle \wp P') \wp P''] \wp P''']]$ and $|\mathcal{P}'| < |\mathcal{P}|$, by the inductive hypothesis, Point 2 applies to \mathcal{P}' . This means there exist $\mathcal{E} : [P_1 \wp P_2] \vdash P'''$, and $\mathcal{P}'' : \vdash [[[\langle R \wp T \rangle \wp P'] \wp P_1]]$, and $\mathcal{Q} : \vdash [P'' \wp P_2]$.

Thanks to $[[[\langle R \wp T \rangle \wp P'] \wp P'']] < [[[(\langle R \wp T \rangle \wp P') \wp P''] \wp P''']]$ the inductive hypothesis holds on \mathcal{P}'' which implies $\mathcal{E}' : \langle U_1 \wp U_2 \rangle \vdash [P' \wp P_1]$, and $\mathcal{Q}_1 : \vdash [R \wp U_1]$, and $\mathcal{Q}_2 : \vdash$

$[T \wp U_2]$. Both \mathcal{Q}_1 , and \mathcal{Q}_2 are the two proofs of BVQ in the statement we have to prove. The derivation is:

$$\begin{array}{c}
\langle U_1 \wp U_2 \rangle \\
\mathcal{E}' \parallel \\
[P' \wp P_1] \\
(15) \frac{}{[(\circ \otimes P') \wp P_1]} \\
\mathcal{Q} \parallel \\
\frac{[(P'' \wp P_2) \otimes P'] \wp P_1}{s \frac{[(P'' \otimes P') \wp P_2] \wp P_1}{(11),(14),(10) \frac{[(P' \otimes P'') \wp [P_1 \wp P_2]]}{\mathcal{E} \parallel [(P' \otimes P'') \wp P''']}}}
\end{array}$$

As a *fifth case* of Point 1 let ρ be $u\downarrow$ with $[\langle R \wp T \rangle \wp P]$ as its redex. This means $P \approx [U]_a$, for some U and a , that, without loss of generality, thanks to (19), we can assume such that $a \in \text{fn}(U)$, and $a \notin \text{fn}(\langle R \wp T \rangle)$. So, by (18), $\langle R \wp T \rangle \approx [\langle R \wp T \rangle]_a$, the derivation is:

$$\begin{array}{c}
\mathcal{P}' \parallel \\
\frac{[[\langle R \wp T \rangle \wp U]]_a}{u\downarrow \frac{}{[[\langle R \wp T \rangle]_a \wp [U]_a]}}
\end{array}$$

Point 3 of Proposition 3.1, applied on \mathcal{P}' , implies:

$$\begin{array}{c}
\mathcal{P}'' \parallel \\
[\langle R \wp T \rangle \wp U]
\end{array}$$

Thanks to $[[\langle R \wp T \rangle \wp U]] < |[[\langle R \wp T \rangle]_a \wp [U]_a]|$ the inductive hypothesis holds on \mathcal{P}'' which implies $\mathcal{E} : \langle P_1 \wp P_2 \rangle \vdash U$, and $\mathcal{Q}_1 : \vdash [R \wp P_1]$, and $\mathcal{Q}_2 : \vdash [T \wp P_2]$. Both \mathcal{Q}_1 , and \mathcal{Q}_2 are the two poofs of BVQ in the statement we have to prove. The derivation is $[\langle P_1 \wp P_2 \rangle]_a \vdash [U]_a$, we obtain from \mathcal{E} thanks to Fact 2.3.

We have exhausted the interesting cases relative to Point 1.

Recall that we prove Point 1, and Point 2 simultaneously, by induction on the lexicographic order $(|U|, |\mathcal{P}|)$, where U is one among $[\langle R \wp T \rangle \wp P]$, and $[(R \otimes T) \wp P]$, proceeding by cases on the last rule ρ of \mathcal{P} . Now we explore the cases relative to Point 2.

As a *first case* of Point 2 let ρ be $q\downarrow$ with $[(R \otimes T) \wp [\langle P' \wp P'' \rangle \wp [U' \wp U'']]]$ as its redex. So, \mathcal{P} can be:

$$\begin{array}{c}
\mathcal{P}' \parallel \\
(16),(10) \frac{[[[(R \otimes T) \wp P'] \wp U'] \wp P''] \wp U'']}{\frac{[[[(R \otimes T) \wp U'] \wp P'] \wp [U' \wp P'']] \wp U''}{q\downarrow \frac{[[[(R \otimes T) \wp U'] \wp U'] \wp [P' \wp P'']] \wp U''}{(14),(10),(16) \frac{}{[(R \otimes T) \wp [\langle P' \wp P'' \rangle \wp [U' \wp U'']]]}}}}
\end{array}$$

Thanks to $[[[(R \otimes T) \wp [\langle P' \wp P'' \rangle \wp [U' \wp U'']]]] = |[[[(R \otimes T) \wp P'] \wp U'] \wp P''] \wp U''||$ and $|\mathcal{P}'| < |\mathcal{P}|$, by the inductive hypothesis, Point 1 applies to \mathcal{P}' . There exist $\mathcal{E} : \langle U_1 \wp U_2 \rangle \vdash U''$, and $\mathcal{P}'' : \vdash [[[(R \otimes T) \wp P'] \wp U'] \wp U_1]$, and $\mathcal{Q} : \vdash [P'' \wp U_2]$.

Thanks to $[[[(R \otimes T) \wp P'] \wp U'] \wp U_1] < |[[[(R \otimes T) \wp P'] \wp U'] \wp P''] \wp U''||$ the inductive hypothesis holds on \mathcal{P}'' which implies $\mathcal{E}' : [P_1 \wp P_2] \vdash [[P' \wp U'] \wp U_1]$, and $\mathcal{Q}_1 : \vdash [R \wp P_1]$, and $\mathcal{Q}_2 : \vdash [T \wp P_2]$. Both \mathcal{Q}_1 , and \mathcal{Q}_2 are the two proofs of BVQ

in the statement we have to prove. The derivation of BVQ in the statement we have to prove is:

$$\begin{array}{c}
[P_1 \wp P_2] \\
\mathcal{E}' \parallel \\
\frac{[[P' \wp U'] \wp U_1]}{(16),(10),(14) \frac{[[P' \wp U_1] \wp U']}{\mathcal{E}' \parallel}} \\
\mathcal{Q} \parallel \\
\frac{[\langle [P' \wp U_1] \wp [P'' \wp U_2] \rangle \wp U']}{\text{ql} \frac{[[\langle P' \wp P'' \rangle \wp \langle U_1 \wp U_2 \rangle] \wp U']}{(10) \frac{[\langle P' \wp P'' \rangle \wp [U' \wp \langle U_1 \wp U_2 \rangle]]}{\mathcal{E}' \parallel}}}} \\
[\langle P' \wp P'' \rangle \wp [U' \wp U'']]
\end{array}$$

As a *second case* of Point 2 let ρ be s with $[(R' \otimes R'') \otimes (T' \otimes T'')] \wp [P' \wp P'']$ as its redex. So, \mathcal{P} can be:

$$\begin{array}{c}
\mathcal{P}' \parallel \\
\frac{[[[(R' \otimes T') \wp P'] \otimes (R'' \otimes T'')] \wp P'']}{\text{s} \frac{[[[(R' \otimes T') \otimes (R'' \otimes T'')] \wp P'] \wp P'']}{(14),(10),(12),(11) \frac{[[[(R' \otimes R'') \otimes (T' \otimes T'')] \wp [P' \wp P'']]}}
\end{array}$$

Both $[[[(R' \otimes R'') \otimes (T' \otimes T'')] \wp [P' \wp P'']] = [[[(R' \otimes T') \wp P'] \otimes (R'' \otimes T'')] \wp P'']$, and $|\mathcal{P}'| < |\mathcal{P}|$ imply that the inductive hypothesis applies to \mathcal{P}' . There exist $\mathcal{E}' : [P_1 \wp P_2] \vdash P''$, and $\mathcal{P}'' : \vdash [[(R' \otimes T') \wp P'] \wp P_1]$, and $\mathcal{Q} : \vdash [(R'' \otimes T'') \wp P_2]$.

Both $[[[(R' \otimes T') \wp P'] \wp P_1]] < [[[(R' \otimes T') \wp P'] \otimes (R'' \otimes T'')] \wp P'']$ the inductive hypothesis holds on \mathcal{P}'' which implies $\mathcal{E}' : [U'_1 \wp U'_2] \vdash [P' \wp P_1]$, and $\mathcal{Q}'_1 : \vdash [R' \wp U'_1]$, and $\mathcal{Q}'_2 : \vdash [T' \wp U'_2]$.

Thanks to $[[[(R'' \otimes T'') \wp P_2]] < [[[(R' \otimes T') \wp P'] \otimes (R'' \otimes T'')] \wp P'']$ the inductive hypothesis holds on \mathcal{Q} which implies $\mathcal{E}'' : [U''_1 \wp U''_2] \vdash P_2$, and $\mathcal{Q}''_1 : \vdash [R'' \wp U''_1]$, and $\mathcal{Q}''_2 : \vdash [T'' \wp U''_2]$.

The derivation and the two proofs of BVQ in the statement we have to prove are:

$$\begin{array}{c}
\frac{[[U'_1 \wp U''_1] \wp [U'_2 \wp U''_2]]}{(14),(10) \frac{[[U'_1 \wp U'_2] \wp [U''_1 \wp U''_2]]}{\mathcal{E}'' \parallel}} \\
\frac{[[U'_1 \wp U'_2] \wp P_2]}{\mathcal{E}' \parallel} \\
(14) \frac{[[P' \wp P_1] \wp P_2]}{\mathcal{E} \parallel} \\
[P' \wp P'']
\end{array}$$

$$\begin{array}{cc}
\frac{\mathcal{Q}''_1 \parallel \frac{[R'' \wp U''_1]}{(15) \frac{[(\circ \otimes R'') \wp U''_1]}{\mathcal{Q}''_1 \parallel}}}{(14),\text{s} \frac{[[[(R' \wp U'_1] \otimes R'') \wp U''_1]]}{[(R' \otimes R'') \wp [U'_1 \wp U''_1]]}} & \frac{\mathcal{Q}''_2 \parallel \frac{[T'' \wp U''_2]}{(15) \frac{[(\circ \otimes T'') \wp U''_2]}{\mathcal{Q}''_2 \parallel}}}{(14),\text{s} \frac{[[[(T' \wp U'_2] \otimes T'') \wp U''_2]]}{[(T' \otimes T'') \wp [U'_2 \wp U''_2]]}}
\end{array}$$

As a *third case* of Point 2 let ρ be \mathbf{s} with $[(R \otimes T) \wp (P' \otimes P'') \wp [U' \wp U'']]$ as its redex. So, \mathcal{P} can be:

$$\frac{\frac{\mathcal{P}' \parallel}{\mathbf{s} \frac{[(P' \wp [(R \otimes T) \wp U']) \otimes P''] \wp U''}{[(P' \otimes P'') \wp [(R \otimes T) \wp U']] \wp U''}}{(14),(10)} \frac{}{[(R \otimes T) \wp (P' \otimes P'') \wp [U' \wp U'']]}$$

Both $||[(P' \wp [(R \otimes T) \wp U']) \otimes P''] \wp U''|| = ||(R \otimes T) \wp [(P' \otimes P'') \wp [U' \wp U'']]||$, and $|\mathcal{P}'| < |\mathcal{P}|$ imply that the inductive hypothesis holds on \mathcal{P}' . So, we have $\mathcal{E} : [P_1 \wp P_2] \vdash U''$, and $\mathcal{P}'' : \vdash [P'' \wp P_2]$, and $\mathcal{Q} : \vdash [[P' \wp [(R \otimes T) \wp U']] \wp P_1]$.

Both $||[(P' \wp [(R \otimes T) \wp U']) \wp P_1]|| < ||[(P' \wp [(R \otimes T) \wp U']) \otimes P''] \wp U''||$, and $|\mathcal{P}'| < |\mathcal{P}|$ imply that the inductive hypothesis holds on \mathcal{Q} . SO, we have $\mathcal{E}' : [U_1 \wp U_2] \vdash [P' \wp [U' \wp P_1]]$, and $\mathcal{Q}' : \vdash [R \wp U_1]$, and $\mathcal{Q}'' : \vdash [T \wp U_2]$.

Both \mathcal{Q}' , and \mathcal{Q}'' are the two proofs of BVQ of the statement we have to prove. The derivation of BVQ is:

$$\frac{\frac{\frac{[U_1 \wp U_2]}{\mathcal{E}' \parallel} \frac{[P' \wp [U' \wp P_1]]}{(15) \frac{}{[(\circ \otimes P') \wp [U' \wp P_1]]}}{\mathcal{P}'' \parallel} \frac{[(P'' \wp P_2) \otimes P'] \wp [U' \wp P_1]}{\mathbf{s} \frac{[(P'' \otimes P') \wp P_2] \wp [U' \wp P_1]}{(14),(10),(11)} \frac{}{[(P' \otimes P'') \wp [U' \wp [P_1 \wp P_2]]}}{\mathcal{E} \parallel} \frac{}{[(P' \otimes P'') \wp [U' \wp U'']]}$$

As a *fourth case* of Point 2 let ρ be $\mathbf{u}\downarrow$ with $[(R \otimes T) \wp P]$ as its redex. This means $P \approx \lceil U \rceil_a$, for some U and a , that, without loss of generality, thanks to (19), we can assume such that $a \notin \text{fn}((R \otimes T))$. So, by (18), $(R \otimes T) \approx \lceil (R \otimes T) \rceil_a$, and \mathcal{P} is:

$$\frac{\mathcal{P}' \parallel}{\mathbf{u}\downarrow} \frac{\lceil [(R \otimes T) \wp U] \rceil_a}{\lceil (R \otimes T) \rceil_a \wp \lceil U \rceil_a}$$

Point 3 of Proposition 3.1, applied on \mathcal{P}' , implies:

$$\frac{\mathcal{P}'' \parallel}{[(R \otimes T) \wp U]}$$

Thanks to $||[(R \otimes T) \wp U]|| < ||\lceil (R \otimes T) \rceil_a \wp \lceil U \rceil_a||$ the inductive hypothesis holds on \mathcal{P}'' which implies $\mathcal{E} : [P_1 \wp P_2] \vdash U$, and $\mathcal{Q}_1 : \vdash [R \wp P_1]$, and $\mathcal{Q}_2 : \vdash [T \wp P_2]$. Both \mathcal{Q}_1 , and \mathcal{Q}_2 are the two poofs of BVQ in the stetement we have to prove. The derivation is $\lceil (P_1 \otimes P_2) \rceil_a \vdash \lceil U \rceil_a$, we obtain from \mathcal{E} thanks to Fact 2.3.

Proof of Point 3. It holds by induction on $(|R|, |\mathcal{P}|)$, proceeding by cases on the last rule ρ of \mathcal{P} .

As a *first case* let the redex of ρ be inside P . So, \mathcal{P} is:

$$\frac{\frac{\mathcal{P}' \parallel}{[R \wp P']}}{\rho} \frac{}{[R \wp P]}$$

We can conclude by applying the inductive hypothesis on \mathcal{P}' , and ρ in the obvious way.

As a *second case*, let ρ be $\text{q}\downarrow$ with $P \approx [\langle P' \circ P'' \rangle \wp P''']$. Also, let R_0 , and R_1 such that $R \approx [R_0 \wp R_1]$. The proof \mathcal{P} can be:

$$\frac{\frac{\frac{\mathcal{P}' \parallel}{\langle [R_1 \wp P'] \circ P'' \rangle \wp [R_0 \wp P''']} \text{q}\downarrow, (14), (16), (10)}{[R_0 \wp [\langle R_1 \circ \circ \rangle \wp \langle P' \circ P'' \rangle] \wp P''']} (14), (16)}{[[R_0 \wp R_1] \wp [\langle P' \circ P'' \rangle \wp P''']]}$$

Point 1 applies to \mathcal{P}' . There are structures P_1, P_2 , such that $\mathcal{E}_0 : \langle P_1 \circ P_2 \rangle \vdash [R_0 \wp P''']$, and $\mathcal{Q}_0 : \vdash [[R_1 \wp P'] \wp P_1] \approx [R_1 \wp [P' \wp P_1]]$, and $\mathcal{Q}_1 : \vdash [P'' \wp P_2]$.

We observe that $|R_1| < |[R_0 \wp R_1]|$. So, the inductive hypothesis holds on \mathcal{Q}_0 . It implies that, for every R_0^1, R_1^1 , if $R_1 \approx [R_0^1 \wp R_1^1]$, then $\mathcal{E}_1 : R_1^1 \vdash [R_0^1 \wp [P' \wp P_1]]$. In particular, it holds $\mathcal{E}'_1 : \overline{R_1} \vdash [\circ \wp [P' \wp P_1]] \approx [P' \wp P_1]$ by taking $R_1 \approx R_1^1$, and $\circ \approx R_0^1$.

We can conclude as follows:

$$\frac{\frac{\frac{\frac{\overline{R_1}}{(16) \overline{\langle R_1 \circ \circ \rangle}} \mathcal{Q}_1 \parallel}{\overline{\langle R_1 \circ [P'' \wp P_2] \rangle}} \mathcal{E}'_1 \parallel}{\text{q}\downarrow \frac{\langle [P' \wp P_1] \circ [P'' \wp P_2] \rangle}{[\langle P' \circ P'' \rangle \wp \langle P_1 \circ P_2 \rangle]}} \mathcal{E}_0 \parallel}{(14), (10) \frac{[\langle P' \circ P'' \rangle \wp [R_0 \wp P''']]}{[R_0 \wp [\langle P' \circ P'' \rangle \wp P''']]}}$$

As a *third case* let ρ be $\text{q}\downarrow$ with $P \approx [\langle P' \circ P'' \rangle \wp [R' \wp R'']]$. So, \mathcal{P} can be:

$$\frac{\frac{\frac{\mathcal{P}' \parallel}{\langle P' \circ [[R_0 \wp R_1] \wp P''] \rangle \wp [R' \wp R'']} \text{q}\downarrow, (16)}{[[\langle \circ \circ [R_0 \wp R_1] \rangle \wp \langle P' \circ P'' \rangle] \wp [R' \wp R'']] (14), (16)}{[[R_0 \wp R_1] \wp [\langle P' \circ P'' \rangle \wp [R' \wp R'']]]}$$

Point 1 applies to \mathcal{P}' . There are structures P_1, P_2 such that there exist $\mathcal{E}_0 : \langle P_1 \circ P_2 \rangle \vdash [R' \wp R'']$, and $\mathcal{Q}_0 : \vdash [P' \wp P_1]$, and $\mathcal{Q}_1 : \vdash [[R_0 \wp R_1] \wp P''] \wp P_2 \approx [R_1 \wp [R_0 \wp [P'' \wp P_2]]]$.

We observe that $|R_1| < |[R_0 \wp R_1]|$. So, the inductive hypothesis holds on \mathcal{Q}_1 . It implies that, for every R_0^1, R_1^1 , if $R_1 \approx [R_0^1 \wp R_1^1]$, then $\mathcal{E}_1 : R_1^1 \vdash [R_0^1 \wp [R_0 \wp [P'' \wp P_2]]]$. In particular, it holds $\mathcal{E}'_1 : \overline{R_1} \vdash [\circ \wp [R_0 \wp [P'' \wp P_2]]] \approx [R_0 \wp [P'' \wp P_2]]$, by taking $R_1 \approx R_1^1$, and $\circ \approx R_0^1$.

We can conclude as follows:

$$\begin{array}{c}
\overline{R_1} \\
(16) \frac{}{\langle \circ \circ \overline{R_1} \rangle} \\
\mathcal{Q}_0 \parallel \\
\langle [P' \wp P_1] \circ \overline{R_1} \rangle \\
\mathcal{E}'_1 \parallel \\
\frac{\langle [P' \wp P_1] \circ [R_0 \wp [P'' \wp P_2]] \rangle}{(14) \frac{\langle [P' \wp P_1] \circ [[R_0 \wp P''] \wp P_2] \rangle}{\text{ql} \frac{[\langle P' \circ [R_0 \wp P''] \rangle \wp \langle P_1 \circ P_2 \rangle]}{\mathcal{E}_0 \parallel} \\
\frac{[\langle P' \circ [R_0 \wp P''] \rangle \wp [R' \wp R'']]}{(17) \frac{[\langle [\circ \wp P'] \circ [R_0 \wp P''] \rangle \wp [R' \wp R'']]}{\text{ql} \frac{[[\langle \circ \circ R_0 \rangle \wp \langle P' \circ P'' \rangle] \wp [R' \wp R'']]}{(16),(14) \frac{[R_0 \wp [\langle P' \circ P'' \rangle \wp [R' \wp R'']]}{}}
\end{array}$$

As a *fourth case* let ρ be \mathbf{s} with $P \approx [(P' \otimes P'') \wp P''']$. Also, let R_0 , and R_1 such that $R \approx [R_0 \wp R_1]$. The proof \mathcal{P} can be:

$$\begin{array}{c}
\mathcal{P}' \parallel \\
\frac{[[R_1 \wp P'] \otimes P''] \wp [R_0 \wp P''']]}{\mathbf{s} \frac{[[R_1 \wp \langle P' \otimes P'' \rangle] \wp [R_0 \wp P''']]}{(14),(10) \frac{[[R_0 \wp R_1] \wp [(P' \otimes P'') \wp P''']]}{}}
\end{array}$$

Point 2 applies to \mathcal{P}' . There are structures P_1, P_2 , such that there exist $\mathcal{E}_0 : [P_1 \wp P_2] \vdash [R_0 \wp P''']$, and $\mathcal{Q}_0 : \vdash [[R_1 \wp P'] \wp P_1]$, and $\mathcal{Q}_1 : \vdash [P'' \wp P_2]$.

We observe that $|R_1| < |[R_0 \wp R_1]|$. So, the inductive hypothesis holds on \mathcal{Q}_0 . It implies that, for every R_0^1, R_1^1 , if $R_1 \approx [R_0^1 \wp R_1^1]$, then $\mathcal{E}_1 : \overline{R_1^1} \vdash [R_0^1 \wp [P' \wp P_1]]$. In particular it holds $\mathcal{E}'_1 : \overline{R_1} \vdash [\circ \wp [P' \wp P_1]] \approx [P' \wp P_1]$ by taking $R_1 \approx R_1^1$, and $\circ \approx R_0^1$. We can conclude as follows:

$$\begin{array}{c}
\overline{R_1} \\
(15) \frac{}{(\circ \otimes \overline{R_1})} \\
\mathcal{Q}_1 \parallel \\
([P'' \wp P_2] \otimes \overline{R_1}) \\
\mathcal{E}'_1 \parallel \\
\frac{([P'' \wp P_2] \otimes [P' \wp P_1])}{\mathbf{s} \frac{[(P'' \otimes [P' \wp P_1]) \wp P_2]}{(14),\mathbf{s},(11) \frac{[(P' \otimes P'') \wp [P_1 \wp P_2]]}{\mathcal{E}_0 \parallel} \\
\frac{[(P' \otimes P'') \wp [R_0 \wp P''']]}{(14),(10) \frac{[R_0 \wp [(P' \otimes P'') \wp P''']]}{}}
\end{array}$$

As a *fifth case* let ρ be $\mathbf{u}\downarrow$ with $P \approx [P']_a$. The proof \mathcal{P} can be:

$$\begin{array}{c}
\mathcal{P}' \parallel \\
\frac{[[R_0 \wp R_1] \wp P']_a}{\mathbf{u}\downarrow \frac{[[R_0 \wp R_1]_a \wp [P']_a]}{(18) \frac{[[R_0 \wp R_1] \wp [P']_a]}{}}}
\end{array}$$

because, thanks to (19), we can always assume P' is such that $a \notin \text{fn}([R_0 \wp R_1])$. Point 3 of Proposition 3.1, applied on \mathcal{P}' , implies:

$$\frac{\mathcal{P}'' \parallel}{[[R_0 \wp R_1] \wp P']}$$

We observe that $|\mathcal{P}''| < |\mathcal{P}|$. So the inductive hypothesis holds on \mathcal{P}'' . It implies that, for every R_0^1, R_1^1 , if $[R_0 \wp R_1] \approx [R_0^1 \wp R_1^1]$, there are $\mathcal{E} : \overline{R_1^1} \vdash [R_0^1 \wp P']$. In particular it holds $\mathcal{E}'_1 : \overline{R_1} \vdash [R_0 \wp P']$ by taking $R_1 \approx R_1^1$, and $R_0 \approx R_0^1$. We can conclude as follows:

$$\begin{array}{c} \overline{R_1} \\ (18) \frac{}{\overline{[R_1]_a}} \\ \mathcal{E}'_1 \parallel \\ \frac{[[R_0 \wp P']]_a}{\text{u}\downarrow} \\ (18) \frac{[[R_0]_a \wp [P']_a]}{[R_0 \wp [P']_a]} \end{array}$$

The topmost instance of (18) is legal thanks to $a \notin \text{fn}([R_0 \wp R_1])$.

Proof of Point 4. The proof is by induction on $|\mathcal{P}|$, proceeding by cases on the last rule ρ of \mathcal{P} .

As a *first case* let the last rule of \mathcal{P} be $\text{q}\downarrow$ with $[[R]_a \wp [\langle P' \wp P'' \rangle \wp [U' \wp U'']]]$ as its redex. So, \mathcal{P} can be:

$$\frac{\mathcal{P}' \parallel}{(14),(16),\text{q}\downarrow,(16) \frac{[\langle P' \wp [R]_a \wp P'' \rangle \wp [U' \wp U'']]}{[[R]_a \wp [\langle P' \wp P'' \rangle \wp [U' \wp U'']]]}}$$

Point 1 applies to \mathcal{P}' . There exist $\mathcal{E} : \langle P_1 \wp P_2 \rangle \vdash [U' \wp U'']$, and $\mathcal{P}'' : \vdash [P' \wp P_1]$, and $\mathcal{Q} : \vdash [[R]_a \wp P'']$. The inductive hypothesis holds on \mathcal{Q} . Thanks to $[[[R]_a \wp P''] \wp P_2] < [[R]_a \wp [\langle P' \wp P'' \rangle \wp [U' \wp U'']]]$ we get $\mathcal{E}' : [U]_a \vdash [P'' \wp P_2]$, and $\mathcal{Q}' : \vdash [R \wp U]$. The proof of BVQ in the statement we have to prove is \mathcal{Q}' . The derivation of BVQ in the statement we have to prove is:

$$\begin{array}{c} [U]_a \\ (16) \frac{}{\langle \circ \wp [U]_a \rangle} \\ \mathcal{P}'' \parallel \\ \langle [P' \wp P_1] \wp [U]_a \rangle \\ \mathcal{E}' \parallel \\ \langle [P' \wp P_1] \wp [P'' \wp P_2] \rangle \\ \text{q}\downarrow \frac{}{[\langle P' \wp P'' \rangle \wp \langle P_1 \wp P_2 \rangle]} \\ \mathcal{E} \parallel \\ [\langle P' \wp P'' \rangle \wp [U' \wp U'']] \end{array}$$

As a *second case* let the last rule of \mathcal{P} be $\text{q}\downarrow$ with $[[R]_a \wp [\langle P' \wp P'' \rangle \wp P''']]$ as its redex. So, \mathcal{P} can be:

$$\frac{\mathcal{P}' \parallel}{(14),(16),\text{q}\downarrow,(17) \frac{[\langle P' \wp [R]_a \wp P'' \rangle \wp P''']}{[[R]_a \wp [\langle P' \wp P'' \rangle \wp P''']]}}$$

Point 1 applies to \mathcal{P}' . There exist $\mathcal{E} : \langle P_1 \wp P_2 \rangle \vdash P'''$, and $\mathcal{P}'' : \vdash [P' \wp P_1]$, and $\mathcal{Q} : \vdash [[R]_a \wp [P'' \wp P_2]]$. Thanks to $[[[R]_a \wp [P'' \wp P_2]] < [[R]_a \wp [\langle P' \wp P'' \rangle \wp P''']]$

the inductive hypothesis holds on \mathcal{Q} which implies $\mathcal{E}' : \lceil U \rceil_a \vdash [P'' \wp P_2]$, and $\mathcal{Q}' : \vdash [R \wp U]$. The proof of BVQ in the statement we have to prove is \mathcal{Q}' . The derivation is:

$$\frac{\frac{\frac{\lceil U \rceil_a}{\mathcal{E}' \parallel} [P'' \wp P_2]}{(16) \frac{\langle \circ \wp [P'' \wp P_2] \rangle}{\mathcal{Q}' \parallel} \frac{\langle [P' \wp P_1] \wp [P'' \wp P_2] \rangle}{\text{q}\downarrow \frac{[\langle P' \wp P'' \rangle \wp \langle P_1 \wp P_2 \rangle]}{\mathcal{E} \parallel} [\langle P' \wp P'' \rangle \wp P''']}}{[\langle P' \wp P'' \rangle \wp P''']}$$

As a *third case* let the last rule of \mathcal{P} be s with $[[R]_a \wp ((P' \otimes P'') \wp P''')]$ as its redex. So, \mathcal{P} can be:

$$\frac{\frac{\mathcal{P}' \parallel}{(14),(10),s,(10)} \frac{[[P' \wp [R]_a] \otimes P'' \wp P''']}{[[R]_a \wp ((P' \otimes P'') \wp P''')]]}{[[R]_a \wp ((P' \otimes P'') \wp P''')]]}$$

Point 2 applies to \mathcal{P}' . There exist $\mathcal{E} : [P_1 \wp P_2] \vdash P'''$, and $\mathcal{P}'' : \vdash [[R]_a \wp P'] \wp P_1$, and $\mathcal{Q} : \vdash [P'' \wp P_2]$. Thanks to $[[R]_a \wp [P' \wp P_1]] < [[R]_a \wp ((P' \otimes P'') \wp P''')]$ the inductive hypothesis holds on \mathcal{P}'' which implies $\mathcal{E}' : \lceil U \rceil_a \vdash [P' \wp P_1]$, and $\mathcal{Q}' : \vdash [R \wp U]$. The proof of BVQ in the statement we have to prove is \mathcal{Q}' . The derivation of BVQ in the statement we have to prove is:

$$\frac{\frac{\frac{\lceil U \rceil_a}{\mathcal{E}' \parallel} [P' \wp P_1]}{(15) \frac{[(\circ \wp P') \wp P_1]}{\mathcal{Q} \parallel} \frac{[[P'' \wp P_2] \otimes P'] \wp P_1}{(11),(14),(10),s \frac{[(P' \otimes P'') \wp [P_1 \wp P_2]]}{\mathcal{E} \parallel} [(P' \otimes P'') \wp P''']}}{[(P' \otimes P'') \wp P''']}$$

As a *fourth case* let the last rule of \mathcal{P} be u \downarrow with $[[R]_a \wp P]$ as its redex. This means $P \approx \lceil U \rceil_a$. So, \mathcal{P} is:

$$\frac{\frac{\mathcal{P}' \parallel}{\text{u}\downarrow} \frac{[[R \wp U]_a]}{[[R]_a \wp \lceil U \rceil_a]}}{[[R]_a \wp P]}$$

Point 3 of Proposition 3.1, applied on \mathcal{P}' , implies the existence of $\mathcal{P}'' : \vdash [R \wp U]$, which is the proof of BVQ in the statement we have to prove. The derivation is $\lceil U \rceil_a \vdash \lceil U \rceil_a$.

C Proof of Context Reduction (Proposition 3.4, page 10)

The proof is by induction on $|S\{\ \}|$, proceeding by cases on the form of $S\{\ \}$.

As a *first case*, let $S\{\ \} \approx \langle S'\{\ \} \wp P \rangle$. So, the assumption is $\mathcal{P} : \vdash \langle S'\{R\} \wp P \rangle$. Point 1 of Proposition 3.1 implies $\mathcal{P}' : \vdash S'\{R\}$, and $\mathcal{P}'' : \vdash P$. Thanks to $|S'\{R\}| < |\langle S'\{R\} \wp P \rangle|$ the inductive hypothesis holds on \mathcal{P}' . There are U , and \vec{b} such that, for every V with $\text{fn}(V) \cap$

$\text{bn}(R) = \emptyset$, both $\mathcal{D} : \llbracket [V \otimes U] \rrbracket_b \vdash S'\{V\}$, and $\mathcal{P}''' : \vdash [R \otimes U]$. The proof \mathcal{P}''' is the one we are looking for. To get the derivation we are looking for, we fix V such that $\text{fn}(V) \cap \text{bn}(R) = \emptyset$. This allows to use \mathcal{D} as follows:

$$\begin{array}{c} \llbracket [V \otimes U] \rrbracket_b \\ \mathcal{D} \parallel \\ S'\{V\} \\ (16) \frac{}{\langle S'\{V\} \circ \circ \rangle} \\ \mathcal{P}'' \parallel \\ \langle S'\{V\} \circ P \rangle \end{array}$$

As a *second case*, let $S\{ \} \approx (S'\{ \} \otimes P)$. So, the assumption is $\mathcal{P} : \vdash (S'\{R\} \otimes P)$. Point 2 of Proposition 3.1 implies $\mathcal{P}' : \vdash S'\{R\}$, and $\mathcal{P}'' : \vdash P$. Thanks to $|S'\{R\}| < |(S'\{R\} \otimes P)|$ the inductive hypothesis holds on \mathcal{P}' . There are U , and \vec{b} such that, for every V with $\text{fn}(V) \cap \text{bn}(R) = \emptyset$, both $\mathcal{D} : \llbracket [V \otimes U] \rrbracket_b \vdash S'\{V\}$, and $\mathcal{P}''' : \vdash [R \otimes U]$. The proof \mathcal{P}''' is the one we are looking for. To get the derivation we are looking for, we fix V such that $\text{fn}(V) \cap \text{bn}(R) = \emptyset$. This allows to use \mathcal{D} as follows:

$$\begin{array}{c} \llbracket [V \otimes U] \rrbracket_b \\ \mathcal{D} \parallel \\ S'\{V\} \\ (15) \frac{}{(S'\{V\} \otimes \circ)} \\ \mathcal{P}'' \parallel \\ (S'\{V\} \otimes P) \end{array}$$

As a *third case*, let $S\{ \} \approx \llbracket S'\{ \} \rrbracket_b$ with $b \in \text{fn}(S'\{ \})$. Otherwise it would be meaningless assuming to have $S\{ \}$ with such a form. So, the assumption is $\mathcal{P} : \vdash \llbracket S'\{R\} \rrbracket_b$. Point 3 of Proposition 3.1 implies $\mathcal{P}' : \vdash S'\{R\}$. So, $|S'\{R\}| < |\llbracket S'\{R\} \rrbracket_b|$ implies the inductive hypothesis holds on \mathcal{P}' . There are U , and \vec{b} such that, for every V with $\text{fn}(V) \cap \text{bn}(R) = \emptyset$, both $\mathcal{D} : \llbracket [V \otimes U] \rrbracket_b \vdash S'\{V\}$, and $\mathcal{P}''' : \vdash [R \otimes U]$. The proof \mathcal{P}''' is the one we are looking for. To get the derivation we are looking for, we fix V such that $\text{fn}(V) \cap \text{bn}(R) = \emptyset$. This allows to use \mathcal{D} as follows:

$$\begin{array}{c} \llbracket [V \otimes U] \rrbracket_b \\ \mathcal{D} \parallel \\ \llbracket S'\{V\} \rrbracket_b \end{array}$$

As a *fourth case*, let $S\{ \} \approx \llbracket \langle S'\{ \} \circ P' \rangle \rrbracket_b$. The assumption is $\mathcal{P} : \vdash \llbracket \langle S'\{R\} \circ P' \rangle \rrbracket_b$. Shallow splitting implies the existence of P_1, P_2 such that $\mathcal{D} : \langle P_1 \circ P_2 \rangle \vdash P$, and $\mathcal{P}_1 : \vdash \llbracket S'\{R\} \rrbracket_b$, and $\mathcal{P}_2 : \vdash \llbracket P' \rrbracket_b$. The relation $|\llbracket S'\{R\} \rrbracket_b| < |\llbracket \langle S'\{R\} \circ P' \rangle \rrbracket_b|$, which holds also thanks to $|P_1| < |P|$, implies the inductive hypothesis holds on \mathcal{P}_1 . There are U , and \vec{b} such that, for every V with $\text{fn}(V) \cap \text{bn}(R) = \emptyset$, both $\mathcal{D}' : \llbracket [V \otimes U] \rrbracket_b \vdash \llbracket S'\{V\} \rrbracket_b$, and $\mathcal{P}''' : \vdash [R \otimes U]$. The proof \mathcal{P}''' is the one we are looking for. To get the derivation we are looking for, we fix V such that $\text{fn}(V) \cap \text{bn}(R) = \emptyset$. This allows to use \mathcal{D}' as follows:

$$\begin{array}{c} \llbracket [V \otimes U] \rrbracket_b \\ \mathcal{D}' \parallel \\ \llbracket S'\{V\} \rrbracket_b \\ (16) \frac{}{\langle \llbracket S'\{V\} \rrbracket_b \circ P_1 \rangle} \\ \mathcal{P}_2 \parallel \\ \langle \llbracket S'\{V\} \rrbracket_b \circ P_1 \rangle \circ \langle P' \circ P_2 \rangle \\ \text{ql} \frac{}{\llbracket \langle S'\{V\} \circ P' \rangle \rrbracket_b} \\ \mathcal{D} \parallel \\ \llbracket \langle S'\{V\} \circ P' \rangle \rrbracket_b \end{array}$$

As a *fifth case*, let $S\{ \} \approx \llbracket (S'\{R\} \otimes P') \rrbracket_b$. The assumption is $\mathcal{P} : \vdash \llbracket (S'\{R\} \otimes P') \rrbracket_b$. Shallow splitting implies the existence of P_1, P_2 such that $\mathcal{D} : \llbracket P_1 \otimes P_2 \rrbracket_b \vdash P$, and $\mathcal{P}_1 : \vdash$

$[S'\{R\} \wp P_1]$, and $\mathcal{P}_2 : \vdash [P' \wp P_2]$. The relation $||[S'\{R\} \wp P_1]|| < ||[(S'\{R\} \otimes P') \wp P]||$, which holds also thanks to $|P_1| < |P|$, implies the inductive hypothesis holds on \mathcal{P}_1 . There are U , and \vec{b} such that, for every V with $\text{fn}(V) \cap \text{bn}(R) = \emptyset$, we have $\mathcal{D}' : \llbracket [V \wp U] \rrbracket_{\vec{b}} \vdash [S'\{V\} \wp P_1]$, and $\mathcal{D}''' : \vdash [R \wp U]$. The proof \mathcal{D}''' is the one we are looking for. To get the derivation we are looking for, we fix V such that $\text{fn}(V) \cap \text{bn}(R) = \emptyset$. This allows to use \mathcal{D}' as follows:

$$\begin{array}{c}
\llbracket [V \wp U] \rrbracket_{\vec{b}} \\
\mathcal{D}' \parallel \\
\frac{[S'\{V\} \wp P_1]}{(15) \frac{(\circ \otimes [S'\{V\} \wp P_1])}{\mathcal{P}_2 \parallel}} \\
\frac{([P' \wp P_2] \otimes [S'\{V\} \wp P_1])}{(11), s \frac{[(S'\{V\} \wp P_1) \otimes P'] \wp P_2}{s \frac{[(S'\{V\} \otimes P') \wp P_1] \wp P_2}{(14) \frac{[(S'\{V\} \otimes P') \wp [P_1 \wp P_2]]}{\mathcal{D}' \parallel}}}} \\
\mathcal{D}' \parallel \\
[(S'\{V\} \otimes P') \wp P]
\end{array}$$

As a *sixth case*, let $S\{ \} \approx \llbracket [S'\{ \}]_a \wp P \rrbracket$ with $a \in \text{bn}(S'\{R\})$. Otherwise, it would be meaningless to assume $S\{ \}$ as such. The assumption is $\mathcal{D} : \vdash \llbracket [S'\{R\}]_a \wp P \rrbracket$. Shallow splitting implies the existence of P' such that $\mathcal{D} : \llbracket [P']_a \vdash P$, and $\mathcal{D}' : \vdash [S'\{R\} \wp P']$. The relation $||[S'\{R\} \wp P']|| < ||\llbracket [S'\{R\}]_a \wp P \rrbracket||$, which holds also because $a \in \text{fn}(S'\{R\})$, implies that the inductive hypothesis on \mathcal{D}' is true. There are U , and \vec{b} such that, for every V with $\text{fn}(V) \cap \text{bn}(R) = \emptyset$, both $\mathcal{D}' : \llbracket [V \wp U] \rrbracket_{\vec{b}} \vdash [S'\{V\} \wp P']$, and $\mathcal{D}''' : \vdash [R \wp U]$. The proof \mathcal{D}''' is the one we are looking for. To get the derivation we are looking for, we fix V such that $\text{fn}(V) \cap \text{bn}(R) = \emptyset$. This allows to use \mathcal{D}' as follows:

$$\begin{array}{c}
\llbracket [V \wp U] \rrbracket_{b_1, \dots, b_n, a} \\
\mathcal{D}' \parallel \\
\frac{\llbracket [S'\{V\} \wp P'] \rrbracket_a}{u! \frac{[\llbracket [S'\{V\}]_a \wp [P']_a \rrbracket]}{\mathcal{D}' \parallel}} \\
\mathcal{D}' \parallel \\
\llbracket [S'\{V\}]_a \wp P \rrbracket
\end{array}$$

As a *seventh case*, let $S\{ \} \approx [S'\{ \} \wp [P]_a]$ with $a \in \text{bn}([P]_a)$. Also, without loss of generality, can always choose a such that $a \notin \text{fn}(S'\{R\})$. The assumption is $\mathcal{D} : \vdash [S'\{R\} \wp [P]_a]$. Shallow splitting implies the existence of P' such that $\mathcal{D} : \llbracket [P']_a \vdash P$, and $\mathcal{D}' : \vdash [S'\{R\} \wp P']$. The relation $||[S'\{R\} \wp P']|| < ||[S'\{R\} \wp [P]_a]||$, which holds also because $a \in \text{bn}([P]_a)$, implies that the inductive hypothesis on \mathcal{D}' is true. There are U , and \vec{b} such that, for every V with $\text{fn}(V) \cap \text{bn}(R) = \emptyset$, both $\mathcal{D}' : \llbracket [V \wp U] \rrbracket_{\vec{b}} \vdash [S'\{V\} \wp P']$, and $\mathcal{D}''' : \vdash [R \wp U]$. The proof \mathcal{D}''' is the one we are looking for. To get the derivation we are looking for, we fix V such that $\text{fn}(V) \cap \text{bn}(R) = \emptyset$. This allows to use \mathcal{D}' as follows:

$$\begin{array}{c}
\llbracket [V \wp U] \rrbracket_{b_1, \dots, b_n, a} \\
\mathcal{D}' \parallel \\
\frac{\llbracket [S'\{V\} \wp P'] \rrbracket_a}{u! \frac{[\llbracket [S'\{V\}]_a \wp [P']_a \rrbracket]}{(18) \frac{[S'\{V\} \wp [P']_a]}{\mathcal{D}' \parallel}}} \\
\mathcal{D}' \parallel \\
[S'\{V\} \wp [P]_a]
\end{array}$$

We remark that (18) applies thanks to $a \notin \text{fn}(S'\{R\})$.

D Proof of *Splitting* (Theorem 3.5, page 10)

We obtain the proof of the three statements by composing Context Reduction (Proposition 3.4), and Shallow Splitting (Proposition 3.2) in this order. We develop the details of Points 1, and 3. The proof of Point 2 is analogous to the one of 1.

Point 1. Context Reduction (Proposition 3.4) applies to \mathcal{P} . So, there are U , and \vec{b} such that, for every V , with $\text{fn}(V) \cap \text{bn}(\langle R \multimap T \rangle) = \emptyset$, there exist $\mathcal{D} : \llbracket [V \wp U] \rrbracket_{\vec{b}} \vdash S\{V\}$, and $\mathcal{Q} : \vdash \llbracket \langle R \multimap T \rangle \wp U \rrbracket$. Shallow Splitting (Proposition 3.2) applies to \mathcal{Q} . So, $\mathcal{E} : \langle K_1 \multimap K_2 \rangle \vdash U$, and $\mathcal{Q}_1 : \vdash [R \wp K_1]$, and $\mathcal{Q}_2 : \vdash [T \wp K_2]$, for some K_1, K_2 . Both \mathcal{Q}_1 , and \mathcal{Q}_2 are the two proofs we are looking for. The derivation is:

$$\begin{array}{c} \llbracket [V \wp \langle K_1 \multimap K_2 \rangle] \rrbracket_{\vec{b}} \\ \mathcal{E} \parallel \\ \llbracket [V \wp U] \rrbracket_{\vec{b}} \\ \mathcal{Q} \parallel \\ S\{V\} \end{array}$$

Point 3. Context Reduction (Proposition 3.4) applies to \mathcal{P} . So, there are U , and \vec{b} such that, for every V with $\text{fn}(V) \cap \text{bn}([R]_a) = \emptyset$, there exist $\mathcal{D} : \llbracket [V \wp U] \rrbracket_{\vec{b}} \vdash S\{V\}$, and $\mathcal{Q} : \vdash \llbracket [R]_a \wp U \rrbracket$. Shallow Splitting (Proposition 3.2) applies to \mathcal{Q} . So, $\mathcal{E} : [K]_a \vdash U$, and $\mathcal{Q}' : \vdash [R \wp K]$, for some K . So, \mathcal{Q}' is the proof we are looking for. The derivation is:

$$\begin{array}{c} \frac{\frac{\frac{\llbracket [V \wp K] \rrbracket_{a, \vec{b}}}{\text{u}\downarrow} \quad \llbracket \llbracket [V]_a \wp [K]_a \rrbracket_{\vec{b}}}{(18) \quad \llbracket [V \wp [K]_a] \rrbracket_{\vec{b}}} \quad \mathcal{E} \parallel}{\llbracket [V \wp U] \rrbracket_{\vec{b}}} \quad \mathcal{Q}' \parallel \\ S\{V\} \end{array}$$

The step (18) applies thanks to the assumption that $\text{fn}(V) \cap \text{bn}([R]_a) = \emptyset$, which implies $a \notin \text{fn}(V)$.

E Proof of *Admissibility of the up fragment* (Theorem 3.6, page 10)

As a *first case* we show that $\text{ai}\uparrow$ is admissible for BVQ. So, we start by assuming:

$$\text{ai}\uparrow \frac{\mathcal{P}' \parallel S(a \otimes \bar{a})}{S\{\circ\}}$$

Point 2 of Splitting (Theorem 3.5) applies to \mathcal{P}' , whose conclusion is $S(a \otimes \bar{a})$. There are K_1, K_2 , and \vec{b} such that, for every V with $\text{fn}(V) \cap \text{bn}((a \otimes \bar{a})) = \emptyset$, there exist $\mathcal{D} : \llbracket [V \wp [K_1 \wp K_2]] \rrbracket_{\vec{b}} \vdash S\{V\}$, and $\mathcal{P}_1 : \vdash [a \wp K_1]$, and $\mathcal{P}_2 : \vdash [\bar{a} \wp K_2]$. Shallow splitting (Proposition 3.2) on \mathcal{P}_1 , and \mathcal{P}_2 implies $\mathcal{E}_1 : \bar{a} \vdash K_1$, and $\mathcal{E}_2 : a \vdash K_2$. To build the following proof with the same conclusion as \mathcal{P} , but without its bottommost instance of $\text{ai}\uparrow$ it is enough to observe that among all the possible instances of V there is \circ , because

$\text{fn}(\circ) \cap \text{bn}((a \otimes \bar{a})) = \emptyset$. So, we can prove:

$$\begin{array}{c}
\frac{(18) \quad \frac{\circ}{\vdash \circ]_b}}{\text{ai} \downarrow \frac{\vdash [\bar{a} \wp a]_b}{\vdash [\bar{a} \wp K_2]_b}} \\
\frac{\mathcal{E}_2 \parallel}{\vdash [\bar{a} \wp K_2]_b} \\
\frac{\mathcal{E}_1 \parallel}{\vdash [K_1 \wp K_2]_b} \\
(15) \quad \frac{\vdash [K_1 \wp K_2]_b}{\vdash [\circ \wp [K_1 \wp K_2]]_b} \\
\mathcal{D}' \parallel \\
S\{\circ\}
\end{array}$$

where \mathcal{D}' is \mathcal{D} with V instantiated as \circ .

As a *second case* we show that $\text{q}\uparrow$ is admissible for BVQ. So, we start by assuming:

$$\frac{\mathcal{D}' \parallel \quad S(\langle R \wp U \rangle \otimes \langle T \wp V \rangle)}{\text{q}\uparrow \quad S(\langle R \otimes T \rangle \wp (U \otimes V))}$$

Point 1 of Splitting (Theorem 3.5) applies to \mathcal{D} — beware, not \mathcal{D}' —, whose conclusion is $S(\langle R \otimes T \rangle \wp (U \otimes V))$. There are K_1, K_2 , and \bar{b} such that, for every V' with $\text{fn}(V') \cap \text{bn}(S(\langle R \otimes T \rangle \wp (U \otimes V))) = \emptyset$, there exist $\mathcal{D} : \vdash [[V' \wp \langle K_1 \wp K_2 \rangle]]_{\bar{b}} \vdash S\{V'\}$, and $\mathcal{P}_1 : \vdash [(R \otimes T) \wp K_1]$, and $\mathcal{P}_2 : \vdash [(U \otimes V) \wp K_2]$. Shallow splitting (Proposition 3.2) on both \mathcal{P}_1 , and \mathcal{P}_2 implies $\mathcal{E} : [K_R \wp K_T] \vdash K_1$, and $\mathcal{Q}_1 : \vdash [R \wp K_R]$, and $\mathcal{Q}_2 : \vdash [T \wp K_T]$, and $\mathcal{E}' : [K_U \wp K_V] \vdash K_2$, and $\mathcal{Q}'_1 : \vdash [U \wp K_U]$, and $\mathcal{Q}'_2 : \vdash [V \wp K_V]$. To build the following proof with the same conclusion as \mathcal{D} , but without its bottommost instance of $\text{q}\uparrow$, it is enough to observe that one of the possible instances of V' is $\langle (R \otimes T) \wp (U \otimes V) \rangle$ because, thanks to (19), we can always assume $\text{fn}(\langle (R \otimes T) \wp (U \otimes V) \rangle) \cap \text{bn}(\langle (R \otimes T) \wp (U \otimes V) \rangle) = \emptyset$:

$$\begin{array}{c}
\frac{(18) \quad \frac{\circ}{\vdash \circ]_{\bar{b}}}}{\mathcal{Q}'_2 \parallel} \\
\frac{\mathcal{Q}'_1 \parallel}{\vdash [V \wp K_V]_{\bar{b}}} \\
\frac{\mathcal{Q}'_1 \parallel}{\vdash [T \wp K_T] \wp [V \wp K_V]_{\bar{b}}} \\
\mathcal{Q}'_1 \parallel \\
\vdash [T \wp K_T] \wp [(U \wp K_U) \otimes V] \wp K_V]_{\bar{b}} \\
\mathcal{Q}_1 \parallel \\
\frac{\text{s}^2 \quad \frac{\vdash [([R \wp K_R] \otimes T) \wp K_T] \wp [([U \wp K_U] \otimes V) \wp K_V]_{\bar{b}}}{\vdash [([R \otimes T) \wp K_R \wp K_T] \wp [(U \otimes V) \wp K_U \wp K_V]_{\bar{b}}}}}{\text{q}\downarrow \quad \frac{\vdash [([R \otimes T) \wp (U \otimes V)] \wp [([K_R \wp K_T] \wp [K_U \wp K_V])]]_{\bar{b}}}{\text{pmix} \quad \frac{\vdash [([R \otimes T) \wp (U \otimes V)] \wp [K_R \wp K_T \wp K_U \wp K_V]]_{\bar{b}}}{\mathcal{E}' \parallel}}} \\
\frac{\mathcal{E}' \parallel}{\vdash [\langle (R \otimes T) \wp (U \otimes V) \rangle \wp [K_R \wp K_T \wp K_2]]_{\bar{b}}} \\
\mathcal{E} \parallel \\
\vdash [\langle (R \otimes T) \wp (U \otimes V) \rangle \wp [K_1 \wp K_2]]_{\bar{b}} \\
\mathcal{D}' \parallel \\
S(\langle R \otimes T \rangle \wp (U \otimes V))
\end{array}$$

where \mathcal{D}' is \mathcal{D} with V' instantiated as $\langle (R \otimes T) \wp (U \otimes V) \rangle$.

As a *third case* we show that $\text{u}\uparrow$ is admissible for BVQ. So, we start by assuming:

$$\frac{\mathcal{D}' \parallel \quad S(\vdash R]_a \otimes \vdash T]_a)}{\text{u}\uparrow \quad S(\vdash (R \otimes T)]_a)}$$

Point 3 of Splitting (Theorem 3.5) applies to \mathcal{P} — beware, not \mathcal{P}' —, whose conclusion is $S[(R \otimes T)]_a$. There is K , and \vec{b} such that, for every V with $\text{fn}(V) \cap \text{bn}(S[(R \otimes T)]_a) = \emptyset$, there exist $\mathcal{D} : \llbracket [V \otimes K] \rrbracket_{a,\vec{b}} \vdash S[V]$, and $\mathcal{P}_1 : \vdash [(R \otimes T) \otimes K]$. Shallow splitting (Proposition 3.2) on \mathcal{P}_1 implies $\mathcal{E} : [K_R \otimes K_T] \vdash K$, and $\mathcal{Q}_1 : \vdash [R \otimes K_R]$, and $\mathcal{Q}_2 : \vdash [T \otimes K_T]$. To build the following proof with the same conclusion as \mathcal{P} , but without its bottommost instance of $\text{u}\uparrow$ it is enough to observe that one of the possible instances of V is $S[(R \otimes T)]_a$ such that $\text{fn}(\llbracket (R \otimes T) \rrbracket_a) \cap \text{bn}(\llbracket (R \otimes T) \rrbracket_a) = \emptyset$:

$$\begin{array}{c}
\circ \\
\hline
(15),(18) \frac{\llbracket (\circ \otimes \circ) \rrbracket_{a,\vec{b}}}{\mathcal{Q}_1 \parallel} \\
\llbracket (\circ \otimes [R \otimes K_R]) \rrbracket_{a,\vec{b}} \\
\mathcal{Q}_2 \parallel \\
\frac{\llbracket ([T \otimes K_T] \otimes [R \otimes K_R]) \rrbracket_{a,\vec{b}}}{(11),s \frac{\llbracket ([R \otimes K_R] \otimes T) \otimes K_T \rrbracket_{a,\vec{b}}}{s \frac{\llbracket [(R \otimes T) \otimes K_R \otimes K_T] \rrbracket_{a,\vec{b}}}{\mathcal{E} \parallel} \\
\llbracket [(R \otimes T) \otimes K] \rrbracket_{a,\vec{b}} \\
\mathcal{D} \parallel \\
S[(R \otimes T)]_a
\end{array}$$

F Proof that $\text{mt}\downarrow$ is derivable in BVQ (Lemma 5.3, page 12)

We proceed by induction on the size $|\langle M \rangle_o|$, of $\langle M \rangle_o$, that occurs in the conclusion of $\text{mt}\downarrow$, proceeding by cases on the form of M .

The first base case is $M \equiv x$.

$$\text{tj} \frac{\langle x \rangle_o \equiv \langle x \circ \bar{o} \rangle}{\llbracket [x]_r \otimes \langle r \circ \bar{o} \rangle \rrbracket \equiv \llbracket \langle x \circ \bar{r} \rangle \otimes \langle r \circ \bar{o} \rangle \rrbracket}$$

The second base case is $M \equiv (M') M''$.

$$\begin{array}{c}
\text{tj} \frac{\langle (M') M'' \rangle_o \equiv \llbracket \llbracket [M']_p \otimes \llbracket [M'']_{q,q} \otimes \langle p \circ \bar{o} \rangle \rrbracket_p \rrbracket}{\llbracket \llbracket [M']_p \otimes \llbracket [M'']_{q,q} \otimes \langle p \circ \bar{r} \rangle \otimes \langle r \circ \bar{o} \rangle \rrbracket_p \rrbracket} \\
(10),\text{u}\downarrow \frac{\llbracket \llbracket [M']_p \otimes \llbracket [M'']_{q,q} \otimes \langle p \circ \bar{r} \rangle \otimes \langle r \circ \bar{o} \rangle \rrbracket_p \rrbracket}{\llbracket \llbracket [M']_p \otimes \llbracket [M'']_{q,q} \otimes \langle p \circ \bar{r} \rangle \rrbracket_p \otimes \langle r \circ \bar{o} \rangle \rrbracket}
\end{array}$$

The unique inductive case is with $M \equiv \lambda y.M'$ that, without loss of generality, can have $y \neq x$.

$$\begin{array}{c}
\text{mt}\downarrow \frac{\langle \lambda y.M' \rangle_o \equiv \llbracket [M']_o \rrbracket_y}{\llbracket \llbracket [M']_r \otimes \langle r \circ \bar{o} \rangle \rrbracket_y \rrbracket} \\
(18),\text{u}\downarrow \frac{\llbracket \llbracket [M']_r \otimes \langle r \circ \bar{o} \rangle \rrbracket_y \rrbracket}{\llbracket \llbracket [M']_r \otimes \langle r \circ \bar{o} \rangle \rrbracket_y \otimes \langle r \circ \bar{o} \rangle \rrbracket}
\end{array}$$

where $\text{mt}\downarrow$ applies by induction because $|\langle M' \rangle_r| < |\langle \lambda y.M' \rangle_r|$.

G Proof that subst is derivable in BVQ (Lemma 5.3, page 12)

We proceed by induction on the size $|\llbracket [M]_o \otimes \langle N \rangle_x \rrbracket|$ of $\llbracket [M]_o \otimes \langle N \rangle_x \rrbracket$, that occurs in the conclusion of subst , proceeding by cases on the form of M .

Let $M \equiv x$. We have three situations:

$N \equiv y$.

$$\text{tl} \frac{\langle x \{^y/x\} \rangle_o \equiv \langle y \rangle_o \equiv \langle y \circ \bar{o} \rangle}{[\langle x \rangle_o \circ \langle y \rangle_x] \equiv [\langle x \circ \bar{o} \rangle \circ \langle y \circ \bar{x} \rangle]}$$

$N \equiv (N') N''$.

$$\begin{array}{c} \text{tl} \frac{\langle x \{^{(N') N''}/x\} \rangle_o \equiv \langle (N') N'' \rangle_o \equiv [\langle N' \rangle_p \circ [\langle N'' \rangle_q \downarrow_q \circ \langle p \circ \bar{o} \rangle]]_p}{[\langle N' \rangle_p \circ [\langle N'' \rangle_q \downarrow_q \circ \langle x \circ \bar{o} \rangle \circ \langle p \circ \bar{x} \rangle]]_p} \\ (14), \text{ul}, (18) \frac{}{[\langle x \rangle_o \circ \langle (N') N'' \rangle_x] \equiv [\langle x \circ \bar{o} \rangle \circ [\langle N' \rangle_p \circ [\langle N'' \rangle_q \downarrow_q \circ \langle p \circ \bar{x} \rangle]]_p]} \end{array}$$

$N \equiv \lambda y. N'$ that, without loss of generality, can be $y \neq x$.

$$\begin{array}{c} \text{mtl} \frac{\langle x \{^{\lambda y. N'}/x\} \rangle_o \equiv \langle \lambda y. N' \rangle_o \equiv [\langle N' \rangle_o \downarrow_y]}{[\langle x \circ \bar{o} \rangle \circ \langle N' \rangle_x]_y} \\ (18), \text{ul} \frac{}{[\langle x \rangle_o \circ \langle \lambda y. N' \rangle_x] \equiv [\langle x \circ \bar{o} \rangle \circ [\langle N' \rangle_x \downarrow_y]]} \end{array}$$

Let $M \equiv \lambda y. M'$ that, without loss of generality, can always be such that $y \neq x$.

$$\begin{array}{c} \text{subst} \frac{\langle \lambda y. M' \{^{N'}/x\} \rangle_o \equiv [\langle M' \{^{N'}/x\} \rangle_o \downarrow_y]}{[\langle M' \rangle_o \circ \langle N' \rangle_x]_y} \\ (18), \text{ul} \frac{}{[\langle \lambda y. M' \rangle_o \circ \langle N \rangle_x] \equiv [\langle M' \rangle_o \downarrow_y \circ \langle N \rangle_x]} \end{array}$$

where subst applies by induction because $|\langle M' \rangle_o \circ \langle N' \rangle_x| < |\langle \lambda y. M' \rangle_o \circ \langle N \rangle_x|$.
Let $M \equiv (M') M''$ with $x \in \text{fv}(M')$.

$$\begin{array}{c} \text{subst} \frac{\langle (M') M'' \{^N/x\} \rangle_o \equiv [\langle M' \{^N/x\} \rangle_p \circ [\langle M'' \rangle_q \downarrow_q \circ \langle p \circ \bar{o} \rangle]]_p}{[\langle M' \rangle_p \circ \langle N \rangle_x] \circ [\langle M'' \rangle_q \downarrow_q \circ \langle p \circ \bar{o} \rangle]]_p} \\ (10), \text{ul}, (14) \frac{}{[\langle (M') M'' \rangle_o \circ \langle N \rangle_x] \equiv [\langle M' \rangle_p \circ [\langle M'' \rangle_q \downarrow_q \circ \langle p \circ \bar{o} \rangle]]_p \circ \langle N \rangle_x]} \end{array}$$

where subst can be applied by induction because $|\langle M' \rangle_p \circ \langle N \rangle_x| < |\langle (M') M'' \rangle_o \circ \langle N \rangle_x|$.
Let $M \equiv (M') M''$ with $x \in \text{fv}(M'')$.

$$\begin{array}{c} \text{subst} \frac{\langle (M') M'' \{^N/x\} \rangle_o \equiv [\langle M' \rangle_p \circ [\langle M'' \{^N/x\} \rangle_q \downarrow_q \circ \langle p \circ \bar{o} \rangle]]_p}{[\langle M' \rangle_p \circ [\langle M'' \rangle_q \downarrow_q \circ \langle N \rangle_x] \circ \langle p \circ \bar{o} \rangle]]_p} \\ (10), \text{ul} \frac{}{[\langle M' \rangle_p \circ [\langle M'' \rangle_q \downarrow_q \circ \langle N \rangle_x] \circ \langle p \circ \bar{o} \rangle]]_p} \\ (10), \text{ul}, (14) \frac{}{[\langle (M') M'' \rangle_o \circ \langle N \rangle_x] \equiv [\langle M' \rangle_p \circ [\langle M'' \rangle_q \downarrow_q \circ \langle p \circ \bar{o} \rangle]]_p \circ \langle N \rangle_x]} \end{array}$$

where subst applies by induction as $|\langle M'' \rangle_q \circ \langle N \rangle_x| < |\langle (M') M'' \rangle_o \circ \langle N \rangle_x|$.